PHY-100C FINAL EXAM SOLUTIONS:

Tuesday, June 08, 2010
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PHYSICS 100C Final Exam, Thursday, June 7, 8AM-11AM

1. A loop of radius R has "glued" linear charge density distribution as a function of azimuthal angle φ, $\rho(\varphi)=\rho_0 \cos^2\varphi$. The loop is spinning with angular frequency ω. Find retarded potentials (Lorentz gauge) A and V at the center of the loop.

$$V(0,t)=\frac{1}{4\pi\varepsilon_0}\cdot\int\frac{\rho(\vec{r},t_R)\cdot d\tau'}{R}=\frac{1}{4\pi\varepsilon_0}\int_0^{2\pi}\frac{\rho_0\cdot\cos^2\varphi\cdot R\cdot d\varphi}{R}=$$

$$=\frac{\rho_0}{4\pi\varepsilon_0}\int_0^{2\pi}\cos^2\varphi\cdot d\varphi=\frac{\rho_0}{4\pi\varepsilon_0}\cdot\pi=\frac{\rho_0}{4\varepsilon_0}$$

Here $\int_0^{2\pi}\cos^2\varphi\cdot d\varphi=\int_0^{2\pi}\frac{1+\cos 2\varphi}{2}=\frac{\varphi}{2}\Big|_0^{2\pi}+\frac{\sin 2\varphi}{4}\Big|_0^{2\pi}=\pi$

$$\vec{A}(0,t)=\frac{\mu_0}{4\pi}\int\frac{\vec{J}(r,t_R)\cdot d\tau'}{R}$$

But $|\vec{J}\cdot d\tau'|=\rho\cdot\omega R$ and $\vec{J}(\varphi)=-\vec{J}(\pi+\varphi)$

therefore opposite sections cancel and $\vec{A}=0$

2. Show that the x-rays (for which frequency much greater than resonance frequencies typically in UV range $\omega \gg \omega_j$) have an index of refraction less than 1, and estimate the angle of total internal reflection for 10 keV x-rays incident on vacuum/metal interface at a grazing incidence. You may assume $N=10^{30}$ "free" electrons per m^3 in a metal.

Eq. 9.170: $n=1+\frac{N_m e^2}{2m\varepsilon_0}\cdot\sum\frac{f_j(\omega_j^2-\omega^2)}{(\omega_j^2-\omega^2)^2+\gamma_j^2\omega^2}$

for $\omega \gg \omega_j$

$$\frac{\omega_j^2-\omega^2}{(\omega_j^2-\omega^2)^2+\gamma_j^2\omega^2}\approx\frac{-\omega^2}{\omega^4+\gamma_j^2\omega^2}\approx-\frac{1}{\omega^2}$$

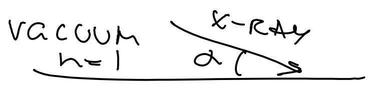
and since $N_m=\frac{\text{molecules}}{\text{volume}}$, $f_j=\frac{\text{electrons}}{\text{molecule}}$

$$N=N_m\cdot\sum f_j\quad\left(\frac{\text{electrons}}{\text{volume}}\right)$$

$$n=1-\frac{Ne^2}{2m\varepsilon_0\cdot\omega^2}=1-\delta$$

where $\delta=\frac{Ne^2}{2m\varepsilon_0\omega^2}=6.8\cdot10^{-6}$

(we use $\omega=\frac{E}{\hbar}=\frac{10\,\text{keV}}{\hbar}=\frac{10^4\cdot e}{\hbar}$ and e^2 cancels)



$n=1-\delta$

SNELL'S LAW:

$$1\cdot\sin\left(\frac{\pi}{2}-\alpha\right)=(1-\delta)\cdot\sin\frac{\pi}{2}$$

OR

$$\cos\alpha\approx1-\frac{\alpha^2}{2}=1-\delta$$

and $\alpha = \sqrt{2\delta} = 3.7 \cdot 10^{-3}$

OR 3,7 mrad OR 2.1 deg

3. a) A loop of radius R (yes, yet another spinning charged loop problem!) is centered at (0,0,0) and is oriented in x-y plane as shown on the Fig. 1. Upper half of the loop has "glued" uniform linear charge density $+\rho$, and bottom half of the loop has linear charge density $-\rho$.
Find total power radiated if the loop is spinning around its axis (with respect to z axis) at angular frequency ω.
b) How would the answer change if the loop is instead is spun around x axis (at the same frequency)? How about if it was spinning around y-axis?

DIPOLE MOMENT (ALONG $\hat{y}$ at $t=0$ due to SYMMETRY)

$$p = 2\int_0^{\pi} \rho \cdot R \cdot \sin\varphi \cdot R \cdot d\varphi = 2\rho R^2(-\cos\varphi)\Big|_0^{\pi} = 4\rho R^2$$

A) DURING ROTATION $\ddot{\vec{p}} = -\omega^2 \vec{p}$

and therefore due to LARMOR FORMULA

$$\text{POWER} = \frac{\mu_0 \cdot |\ddot{p}|^2}{6\pi c} = \frac{\mu_0 \omega^4 (4\rho R^2)^2}{6\pi c} = \frac{8\mu_0 \omega^4 \rho^2 R^4}{3\pi c}$$

B) SPINNING around x Gives the SAME ANSWER AS A) since $\vec{p}$ rotates at ω WITH RESPECT TO AXIS $\perp \vec{p}$.

SPIN AROUND y-axis has $\ddot{p} = 0$ therefore no dipole radiation.

1. a) Electric and magnetic fields at some region of space are given by two vectors, **E** and **B**, such as **E** $\perp$ **B.** Show that there always exists a frame of reference in which **B**=0, and find its velocity (direction and magnitude), as well as the resulting electric field **E'** in this frame of reference.
b) Show that for two arbitrary (no longer mutually perpendicular) vectors **E** and **B**, their scalar product **E·B** is relativistically-invariant. Comment on existence of frame of reference found in a) when **E** and **B** are not perpendicular.

A) LETS DEFINE $\hat{y} \parallel \vec{B}$, $\hat{z} \parallel \vec{E}$

$\hat{x} \perp \vec{E}, \vec{B}$, so that $B_z = 0$, $E_y = 0$

and $E_x = B_x = 0$, $B_y = |\vec{B}|$ $E_z = |\vec{E}|$

In FRAME OF REFERENCE moving along $\hat{x}$:

$E'_x = B'_x = 0$

$E'_y = 0$ $\qquad B'_y = \gamma\left(B_y + \frac{v}{c^2} E_z\right)$

$$E_z' = \gamma(E_z + vB_y) \quad B_z' = 0$$

(Eq. 12.108)

FOR $\bar{B}=0$ (OR $B_y = 0$)

$$B_y + \frac{v}{c^2}\cdot E_z = 0 \Rightarrow v = -\frac{B_y\cdot c^2}{E_z}$$

OR $v = -\frac{|B|\cdot c^2}{|E|}$

then $|E| = E_z' = \gamma(E_z + vB_y) = \gamma\left(|E| - \frac{|B|^2\cdot c^2}{|E|}\right) =$

$= \gamma\,|E|\cdot\left(1 - \frac{|B|^2\cdot c^2}{|E|^2}\right)$ and

$\gamma = \left[1 - (v/c)^2\right]^{-1/2} = \left(1 - \frac{|B|^2\cdot c^2}{|E|^2}\right)^{-1/2}$, so that

$$|E| = E_z' = |E|\cdot\left(1 - \frac{|B|^2\cdot c^2}{|E|^2}\right)^{1/2}$$

b) $\bar{E}'\cdot\bar{B}' = E_x'\cdot B_x' + E_y'\cdot B_y' + E_z'\cdot B_z'$

USING EQ. 12.108:

$$\bar{E}'\cdot\bar{B}' = E_x\cdot B_x + \gamma^2(E_y - vB_z)\left(B_y + \frac{v}{c^2}E_z\right) +$$

$$+\gamma^2(E_z + vB_y)\cdot\left(B_z - \frac{v}{c^2}E_y\right) = E_x\cdot B_x +$$

$$+\gamma^2\Big(E_yB_y - \cancel{vB_yB_z} + \cancel{\frac{v}{c^2}E_yE_z} - \frac{v^2}{c^2}B_zE_z +$$

$$+ E_zB_z + \cancel{vB_yB_z} - \cancel{\frac{v}{c^2}E_yE_z} - \frac{v^2}{c^2}E_yB_y\Big) =$$

$$= E_xB_x + \underbrace{\gamma^2\cdot\left(1 - \frac{v^2}{c^2}\right)}_{1}\cdot(E_yB_y + E_zB_z) =$$

$$= E_xB_x + E_y\cdot B_y + E_z\cdot B_z = \bar{E}\cdot\bar{B}$$

If $\vec{E}$ is not $\perp$ $\vec{B}$ $\quad \bar{E}\cdot\bar{B} \neq 0$

AND $\bar{E}'\cdot\bar{B}' \neq 0$ so that $B' \neq 0$

in ALL FRAMES

5. Large Hadron Collider is set to collide two beams of protons travelling at the same speed in opposite directions with energy of 7 TeV per proton (in the laboratory frame). What is the energy of on-coming protons in the frame of reference of one of the beams?

$m_0c^2 = 1\,\text{GeV} \qquad E = \gamma m_0 c^2 = 7\,\text{TeV}$

$p_x = -\gamma m_0 v \qquad p_0 = \frac{E}{c}$

$p_0' = \frac{E'}{c} = \gamma(p_0 - \beta\cdot p_x) = \gamma\left(\frac{E}{c} + \beta\gamma m_0 v\right)$

$E' = \gamma E + \gamma^2\cdot m_0 v^2 = \gamma(E + \underbrace{\gamma m_0 c^2}_{E}) + \gamma^2 m_0 \underbrace{\left(\frac{v^2}{c^2} - 1\right)}_{-1/\gamma^2}\cdot c^2$

since $\gamma = \frac{E}{m_0c^2}$

$E' = 2\gamma E - m_0c^2 = \frac{2E^2}{m_0c^2} - m_0c^2$

(ALSO: $E' = (2\gamma^2 - 1)\cdot m_0c^2$ or $E' = \frac{2\gamma^2 - 1}{\gamma}\cdot E$)

In any case, since $\gamma = 7\cdot 10^3$,

$E' = 1.4\cdot 10^4 E$ or $9.8\cdot 10^{16}\,\text{eV}$ or $98\,\text{PeV}$

Peta