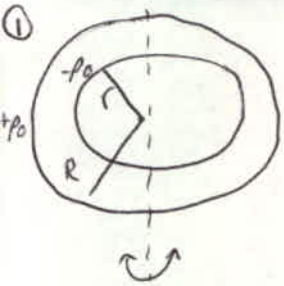
PHYS 100C, Final Exam
June 11, 2012, 11:30AM-2PM, York 4080A

1. Two concentric loops with radii R and r have "glued" charge density of ρ_0 (for large, outside loop with radius R) and $-\rho_0$ (for smaller, inside loop of radius r). Both loops begin to rotate together in an oscillating fashion, so that each point on the loops undergoes periodic motion with angular displacement $\phi = \phi_0 \cos(\omega t)$, where $\phi_0 < 2\pi$.
   Find the (retarded) potentials V and $\vec{\mathbf{A}}$ in the center of the two loops as a function of time t.

2. a) Find the group velocity of a wave in a medium at the resonance condition, $\omega=\omega_0$, ignoring all other resonances.
   b) Under what conditions will the group velocity be negative?

3. A relativistic electron with $\gamma_0 \gg 1$ enters a "booster" linear accelerator (linac), a device of length L with uniform electric field that accelerates the electron to γ_1 ($\gamma_1 > \gamma_0$) at the exit end of the linac.
   a) Find the total energy radiated by an electron during this "boost". You may assume that the particle is always relativistic (v approaching c) and that the particle energy is increased linearly as a function of time and distance of propagation across entire the linac. (Hint: Use the Liénard formula for power, and calculate the acceleration using relativistic momentum).
   b) If the electron with γ_1 then enters a circular storage ring with radius R, find the energy loss due to radiation as the electron completes one lap around the storage ring, assuming this energy loss is much smaller than the total energy of the electron.

① 

$\phi = \phi_0 \cos\omega t$

$\Omega = \frac{d\phi}{dt} = -\omega\phi_0 \sin\omega t$

"observed current" $I = \underset{\substack{\uparrow \\ \text{charge density}}}{\lambda} \overset{=(\text{distance})\Omega}{v} = -\rho_0 r(-\omega\phi_0 \sin\omega t) =$ $\rho_0 r\omega\phi_0 \sin\omega t$ (small loop)

$-\rho_0 R\omega\phi_0 \sin\omega t$ (large loop)

Griffiths (10.19) $V(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{r}',t_r)}{\mathscr{r}}d\tau'$

here we have a linear charge density

$d\tau'$ gets replaced with $dl = r\,d\phi$

$$V = \frac{1}{4\pi\epsilon_0}\left[\overbrace{\rho_0 \cdot 2\pi R}^{\text{total charge}} / \underbrace{R}_{\text{distance}} - \rho_0 \cdot \frac{2\pi r}{r}\right] = \boxed{0 = V}$$

(measured at center of loops)

$\vec{A}(\vec{r}',t) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{r}',t_r)}{\mathscr{r}}d\tau'$

This is a vector integral. current always points in the $\hat{\phi}$ (or $-\hat{\phi}$) direction, so integrated about a closed loop gives zero contribution to $\vec{A}$

$\boxed{\vec{A} = 0}$

② a) $v_g = \frac{d\omega}{dk}$

using Griffiths eqn (9.169)

$$\tilde{k} \approx \frac{\omega}{c}\left[1 + \frac{Nq^2}{2m\epsilon_0} \sum_j \frac{f_j}{\omega_j^2 - \omega^2 - i\gamma_j\omega}\right]$$

we are ignoring all resonances other than $\omega = \omega_0$

$$\tilde{k} \approx \frac{\omega}{c}\left[1 + \frac{Nq^2}{2m\epsilon_0} \frac{f}{\omega_0^2 - \omega^2 - i\gamma\omega}\right]$$ where f is the # electrons

$$\frac{d\tilde{k}}{d\omega} = \frac{1}{c}\left[1 + \frac{Nq^2 f}{2m\epsilon_0} \frac{1}{\omega_0^2 - \omega^2 - i\gamma\omega}\right] + \frac{\omega}{c}\left[-\frac{Nq^2 f}{2m\epsilon_0} \frac{1}{(\omega_0^2 - \omega^2 - i\gamma\omega)^2}(-2\omega - i\gamma)\right]$$

$$\left.\frac{d\tilde{k}}{d\omega}\right|_{\omega=\omega_0} = \frac{1}{c}\left[1 + \cancel{\frac{Nq^2 f}{2m\epsilon_0} \cdot \frac{1}{-i\gamma\omega_0}}\right] + \frac{\omega_0}{c}\left[\frac{Nq^2 f}{2m\epsilon_0} \cdot \frac{-1}{\gamma^2\omega_0^2}(2\omega_0 + \cancel{i\gamma})\right]$$

resonance condition

now just look at the real part (this is what we need for phase velocity) and the imaginary parts cancel anyway!

$$\frac{dk}{d\omega} = \frac{1}{c} + \frac{\cancel{\omega_0}}{c}\left(\frac{-Nq^2 f \cdot \cancel{2}\cancel{\omega_0}}{\cancel{2}m\epsilon_0\gamma^2\cancel{\omega_0^2}}\right) = \frac{1}{c}\left(1 - \frac{Nq^2 f}{m\epsilon_0\gamma^2}\right)$$

$$v_g = \frac{d\omega}{dk} = \frac{1}{\frac{dk}{d\omega}} = \boxed{\frac{c}{1 - \frac{Nq^2 f}{m\epsilon_0\gamma^2}} = v_g}$$

b) this is negative where $\boxed{\frac{Nq^2 f}{m\epsilon_0\gamma^2} > 1}$

③ a) Liénard generalization of the Larmor formula

$$P = \frac{\mu_0 q^2 \gamma^6}{6\pi c}\left(a^2 - \left|\frac{\vec{v}\times\vec{a}}{c}\right|^2\right)$$

in the case of a LINAC, $\vec{v} \parallel \vec{a}$ so $\vec{v}\times\vec{a} = 0$

$$P = \frac{\mu_0 q^2}{6\pi c}\gamma^6 a^2$$

constant force provided by electric field $F = qE_0$
use relativistic momentum

$$F = \frac{dp}{dt} = \frac{d}{dt}(\gamma m_0 v)$$

$$\frac{dp}{dt} = m_0(\dot{\gamma}v + \gamma\dot{v}) \quad \text{where} \quad \gamma = (1 - v^2/c^2)^{-1/2}$$

$$\dot{\gamma} = -\tfrac{1}{2}(1 - v^2/c^2)^{-3/2}(-2v/c^2)\dot{v}$$

$$\dot{\gamma} = \frac{v}{c^2}\gamma^3\dot{v}$$

$$\frac{dp}{dt} = m_0\left(\frac{v^2}{c^2}\gamma^3\dot{v} + \gamma\dot{v}\right) \quad \text{while} \quad \gamma = \frac{1}{\sqrt{1 - v^2/c^2}} \Rightarrow \frac{v^2}{c^2} = \frac{\gamma^2 - 1}{\gamma^2}$$

$$\frac{dp}{dt} = m_0\left(\left(\frac{\gamma^2-1}{\gamma^2}\right)\gamma^3\dot{v} + \gamma\dot{v}\right)$$

$$\frac{dp}{dt} = m_0\left[(\gamma^3 - \cancel{\gamma})\dot{v} + \cancel{\gamma\dot{v}}\right]$$

$$\frac{dp}{dt} = m_0\gamma^3\dot{v} = m_0\frac{c^2}{v}\dot{\gamma}$$ (using formula for $\dot{\gamma}$ derived above)

acceleration $a = \dot{v}$ and $v \sim c$

$$\frac{dp}{dt} = m_0\gamma^3 a = m_0 c\dot{\gamma} = \text{const}$$

$$\gamma^3 a = c\dot{\gamma} = \text{const}$$

$$P = \frac{\mu_0 q^2}{6\pi c}\gamma^6 a^2 = \frac{\mu_0 q^2}{6\pi c}c^2\dot{\gamma}^2 = \frac{\mu_0 q^2 c}{6\pi}\dot{\gamma}^2$$

$$\dot{\gamma} = \frac{d\gamma}{dt} = \frac{\Delta\gamma}{\Delta t} = \frac{\Delta\gamma}{\Delta l}\frac{\Delta l}{\Delta t} = \frac{\Delta\gamma}{\Delta l}v \sim \frac{\Delta\gamma}{\Delta l}c$$

(since $\dot{\gamma}$ = const) length of LINAC

$$P = \frac{\mu_0 q^2 c}{6\pi}\dot{\gamma}^2 = \frac{\mu_0 q^2 c}{6\pi}\left(\frac{\Delta\gamma}{\Delta l}\right)c\left(\frac{\Delta\gamma}{\Delta t}\right)$$

$$\text{Energy} = P\Delta t = \frac{\mu_0 q^2 c^2}{6\pi}\left(\frac{\Delta\gamma}{\Delta l}\right)\Delta\gamma$$

given values in problem, $\Delta\gamma = \gamma_1 - \gamma_0$
$\Delta l = L$

$$\boxed{\text{Energy} = \frac{\mu_0 q^2 c^2(\gamma_1 - \gamma_0)^2}{6\pi L}}$$

③b) Going back to the Liénard formula

$$P = \frac{\mu_0 q^2 \gamma^6}{6\pi c}\left(a^2 - \left|\frac{\vec{v}\times\vec{a}}{c}\right|^2\right)$$

for a circular accelerater (storage ring), $\vec{v} \perp \vec{a}$ so $|\vec{v}\times\vec{a}| = va$

$$P = \frac{\mu_0 q^2 \gamma^6}{6\pi c}\left(a^2 - \frac{v^2 a^2}{c^2}\right) = \frac{\mu_0 q^2 \gamma^6}{6\pi c} a^2\left(1 - \frac{v^2}{c^2}\right)$$

since $\gamma = \frac{1}{\sqrt{1-v^2/c^2}}$, $(1 - v^2/c^2) = \frac{1}{\gamma^2}$

$$P = \frac{\mu_0 q^2 \gamma^6}{6\pi c} a^2 \cdot \frac{1}{\gamma^2} = \frac{\mu_0 q^2 \gamma^4}{6\pi c} a^2$$

all acceleration is centripetal

$a = \frac{\gamma v^2}{R}$ (factor of γ because R is length contracted since the electron is moving relativistically)

$$P = \frac{\mu_0 q^2 \gamma^4}{6\pi c}\left(\frac{\gamma v^2}{R}\right)^2 = \frac{\mu_0 q^2 \gamma^6 v^4}{6\pi c R^2}$$

time to complete one lap around storage ring $\Delta t = \frac{2\pi R}{\gamma v}$

length contracted circumference

since energy lost is small compared to total energy, we can assume that v (and also γ) are constant $\Rightarrow \gamma = \gamma_1$

$$\therefore \text{Energy} = P\Delta t = \frac{\mu_0 q^2 \gamma_1^6 \cancel{v^4}^{3}}{_3\cancel{6\pi} c R^{\cancel{2}}} \cdot \frac{\cancel{2\pi R}}{\gamma \cancel{v}} = \frac{\mu_0 q^2 \gamma_1^5 v^3}{3cR}$$

since $\gamma = \frac{1}{\sqrt{1-v^2/c^2}} \Rightarrow \frac{v^2}{c^2} = \frac{\gamma^2-1}{\gamma^2} \Rightarrow v = \frac{c}{\gamma}\sqrt{\gamma^2-1}$

$$\text{Energy} = \frac{\mu_0 q^2 \gamma_1^{\cancel{5}\,2}}{3\cancel{c}R}\,\frac{c^{\cancel{3}\,2}}{\cancel{\gamma_1^3}}(\gamma^2-1)^{3/2}$$

$$\boxed{\text{Energy} = \frac{\mu_0 q^2 c^2 \gamma_1^2}{3R}(\gamma^2-1)^{3/2}}$$

or, since $v \sim c$ $(\gamma \gg 1)$

$$\boxed{\text{Energy} \approx \frac{\mu_0 q^2 c^2 \gamma_1^5}{3R}}$$