PHYS 100C, LECTURE #6

Thursday, April 16, 2009
6:43 PM

* Waveguides (Cont'd from Lecture #5)

Perfect Conductor:

$E=0$, $B=0$ inside

Boundary conditions:

$$\epsilon_1 E_1^{\perp} - \epsilon_2 E_2^{\perp} = \sigma \qquad \text{(i)}$$

$$B_1^{\perp} = B_2^{\perp} \qquad \text{(ii)}$$

$$E_1^{\parallel} = E_2^{\parallel} \qquad \text{(iii)}$$

$$\frac{1}{\mu_1} B_1^{\parallel} - \frac{1}{\mu_2} B_2^{\parallel} = K_f \times n \qquad \text{(iv)}$$

Or $E^{\parallel} = 0$, $B^{\perp} = 0$ (ii), (iii)

(since $E=0$, $B=0$ inside)

In interior of the waveguide

$$\bar{\nabla} \cdot \bar{E} = 0 \qquad \bar{\nabla} \times \bar{E} = -\frac{\partial B}{\partial t}$$

$$\bar{\nabla} \cdot \bar{B} = 0 \qquad \nabla \times B = \frac{1}{c^2} \frac{\partial E}{\partial t}$$

No free charges, currents, assume vacuum (thus $1/c^2$)

$$\tilde{E} = \tilde{E}_0 \cdot e^{i(kz - \omega t)}$$

$$\tilde{B} = \tilde{B}_0 \cdot e^{i(kz - \omega t)}$$

E, B are not generally transverse (which was a result of plane waves with no x-y dependence for E, B)

$$\nabla\times E = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ E_x & E_y & E_z \end{vmatrix} = \hat{x}\left(\frac{\partial E_z}{\partial y}-\frac{\partial E_y}{\partial z}\right)+$$

$$+\hat{y}\left(\frac{\partial E_x}{\partial z}-\frac{\partial E_z}{\partial x}\right)+$$

$$+\hat{z}\left(\frac{\partial E_y}{\partial x}-\frac{\partial E_x}{\partial y}\right)$$

For (iii):

$$(z):\ \frac{\partial E_y}{\partial x}-\frac{\partial E_x}{\partial y}=i\omega B_z \qquad (1)$$

$$(x):\ \frac{\partial E_z}{\partial y}-ikE_y=i\omega B_x \qquad (2)$$

$$(y):\ ikE_x-\frac{\partial E_z}{\partial x}=i\omega B_y \qquad (3)$$

$$\left(\text{Since } \frac{\partial E_y}{\partial z}=ikE_y \quad \frac{\partial E_x}{\partial z}=ikE_x\right)$$

Similarly for Eq.(iV):

$$(z):\ \frac{\partial B_y}{\partial x}-\frac{\partial B_x}{\partial y}=-\frac{i\omega}{c^2}E_z \qquad (4)$$

$$(x):\ \frac{\partial B_z}{\partial y}-ikB_y=-\frac{i\omega}{c^2}E_x \qquad (5)$$

$$(y):\ ikB_x-\frac{\partial B_z}{\partial x}=-\frac{i\omega}{c^2}E_y \qquad (6)$$

From (3) & (5), eliminate B_y:

$$E_x=\frac{i}{(\omega/c)^2-k^2}\left(k\frac{\partial E_z}{\partial x}+\omega\frac{\partial B_z}{\partial y}\right)$$

From (2) & (6), eliminate B_x

$$E_y=\frac{i}{(\omega/c)^2-k^2}\left(k\frac{\partial E_z}{\partial y}-\omega\frac{\partial B_z}{\partial x}\right)$$

Similarly for B_x, B_y:

(2) and (6):

$$B_x = \frac{i}{(\omega/c)^2 - k^2} \cdot \left(k \frac{\partial B_z}{\partial x} - \frac{\omega}{c^2} \frac{\partial E_z}{\partial y} \right)$$

$$B_y = \frac{i}{(\omega/c)^2 - k^2} \left(k \frac{\partial B_z}{\partial y} + \frac{\omega}{c^2} \frac{\partial E_z}{\partial x} \right)$$

E_z & B_z determine everything else.

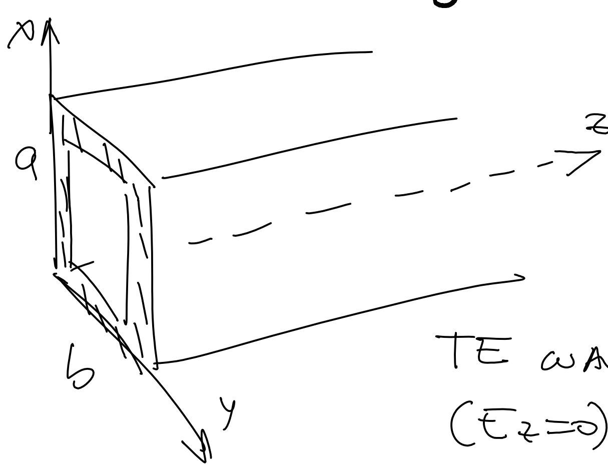
$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \left(\frac{\omega}{c}\right)^2 - k^2 \right) E_z, B_z = 0$$

$E_z = 0 \Rightarrow$ TE waves

$B_z = 0 \Rightarrow$ TM waves

$E_z = B_z = 0 \Rightarrow$ TEM waves (cannot occur in hollow WG)

* Rectangular Waveguide:



TE waves ($E_z = 0$)

$$E_z, B_z(x,y) = X(x) Y(y)$$

$$Y\frac{\partial^2 X}{\partial x^2}+X\frac{\partial^2 Y}{\partial y^2}+\left[\left(\frac{\omega}{c}\right)^2-k^2\right]XY=0$$

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2}=-k_x^2 \qquad \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2}=-k_y^2$$

$$k_x^2+k_y^2=\left(\frac{\omega}{c}\right)^2-k^2$$

$$X=A\sin(k_x\cdot x)+B\cdot\cos(k_x\cdot x)$$

at $x=0$; $x=a$ $\quad B_x=0$

$$B_x=\ldots\underbrace{\frac{\partial B_z}{\partial x}}_{\frac{\partial X}{\partial x}\cdot Y=0}-\ldots\underbrace{\frac{\partial E_z}{\partial y}}_{0\ (TE)}=0$$

$$\frac{\partial X}{\partial x}=k_x\left(A\cdot\cos(k_x\cdot x)-B\sin(k_x\cdot x)\right)=0$$

for $x=0,\ a \Rightarrow A=0$

$\sin(k_x\cdot x)=0$ at $x=a$

$$k_x=\frac{m\pi}{a} \qquad m=0,1,2\ldots$$

Similarly $k_y=\dfrac{n\pi}{b}$ $\quad n=0,1,2\ldots$

$$B_z=XY=B_0\cdot\cos\frac{m\pi x}{a}\cdot\cos\frac{n\pi x}{b}$$

TE_{mn} mode

Standing waves (2D) in xy

$$k=\sqrt{\left(\frac{\omega}{c}\right)^2-\left(\frac{\pi m}{a}\right)^2-\left(\frac{\pi n}{b}\right)^2}$$

$$\omega_{mn}^2 = \left(\frac{\pi m c}{a}\right)^2 + \left(\frac{\pi n c}{b}\right)^2$$

cut-off frequency for TE_{mn}

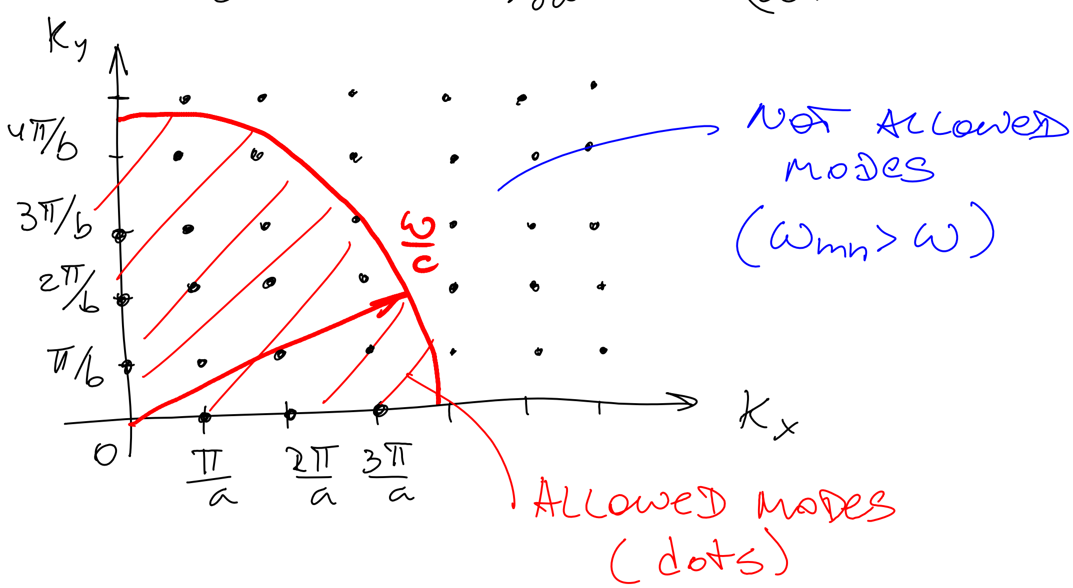
If $\omega < \omega_{mn}$, k is complex (attenuation)

$$k = \frac{1}{c}\sqrt{\omega^2 - \omega_{mn}^2}$$

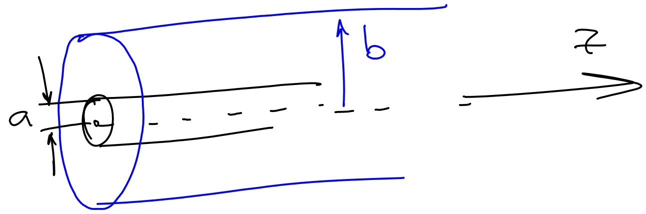
$$v = \frac{\omega}{k} = \frac{c}{\sqrt{1-\left(\frac{\omega_{mn}}{\omega}\right)^2}} \quad (>c)$$

Group velocity:

$$v_g = \frac{\partial \omega}{\partial k} = \frac{1}{\partial k / \partial \omega} = c\sqrt{1-\left(\frac{\omega_{mn}}{\omega}\right)^2} < c$$



* **Coaxial cable:** (NOT COVERED IN CLASS)



SOLUTION ALLOWS TEM
$(B_z = E_z = 0)$

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = i\omega B_z = 0$$

$$\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} = -\frac{i\omega}{c^2} E_z = 0$$

$$\nabla \cdot E = 0 \Rightarrow \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} = 0$$

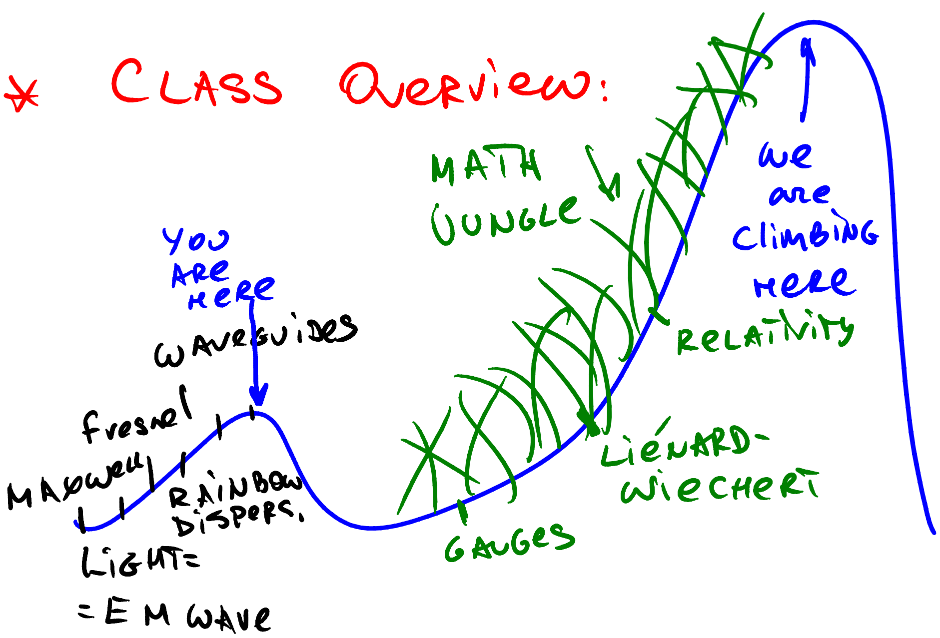
$$\nabla \cdot B = 0 \Rightarrow \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} = 0$$

Electro/Magneto-statics:

$$\vec{E} = \frac{A \cdot \cos(kz - \omega t)}{r} \cdot \hat{r}$$

$$\vec{B} = \frac{A \cdot \cos(kz - \omega t)}{cr} \cdot \hat{\phi}$$

* CLASS Overview:



* Potential & Fields

In electrostatics:

$$\nabla \cdot E = \frac{\rho}{\varepsilon_0} \qquad \nabla \times E = -\frac{\partial B}{\partial t} = 0 \quad (B = \text{const})$$

This allowed introducing scalar potential V, This allowed introducing scalar potential V,

$$\vec{E} = -\vec{\nabla} V$$

Since $\vec{\nabla} \times \vec{E} = -\vec{\nabla} \times (\vec{\nabla} V) = 0$

No longer works if $\partial B / \partial t \neq 0$

$\vec{\nabla} \cdot \vec{B} = 0$ so vector potential still works:

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

then $\nabla \times E = -\frac{\partial B}{\partial t}$ becomes:

$$\nabla \times E = -\frac{\partial}{\partial t}(\nabla \times A) \quad \text{or}$$

$$\nabla \times \left(E + \frac{\partial A}{\partial t}\right) = 0$$

New quantity, $E' = E + \frac{\partial A}{\partial t} = -\nabla V$

since $\nabla \times E' = 0$

then $E = -\nabla V - \frac{\partial A}{\partial t}$

Gauss' Law: $\nabla^2 V + \underbrace{\frac{\partial}{\partial t}(\nabla \cdot A)} = -\frac{\rho}{\varepsilon_0}$

similar to Poisson Eq. EXCEPT for this

last Maxwell Eq:

$$\left(\nabla^2 A - \mu_0 \varepsilon_0 \frac{\partial^2 A}{\partial t^2}\right) - \nabla\left(\vec{\nabla} \cdot \vec{A} + \mu_0 \varepsilon_0 \frac{\partial V}{\partial t}\right) = -\mu_0 J$$

Monstrous, UGLY equation.

But, reduced 6 unknowns $\begin{pmatrix} E_x & E_y & E_z \\ B_x & B_y & B_z \end{pmatrix}$

to only 4 (V, A_x, A_y, A_z).

Useful for other purposes.

Useful for other purposes.

A & V not unique:

can introduce new

$$A' = A + \alpha$$
$$V' = V + \beta$$

that give rise to the same $\vec{E}(\vec{r},t)$ $\vec{B}(\vec{r},t)$

Called GAUGE TRANSFORMATIONS

TO BE CONTINUED...