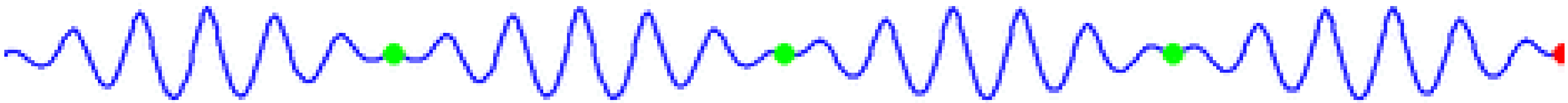


- Phase velocity

$$V_p = \frac{\omega}{k} = \lambda \cdot f$$

- Group velocity

$$V_g = \frac{\partial \omega}{\partial k} = 2\pi \frac{\partial f}{\partial k} = \frac{2\pi}{2\pi} \cdot h \cdot \frac{\partial f}{\partial p} = v$$

$$f = \frac{E}{h} = \frac{p^2}{2m} \cdot \frac{1}{h} \qquad \lambda = \frac{h}{p} \qquad k = \frac{2\pi}{\lambda} = \frac{2\pi p}{h}$$

$$\frac{\partial f}{\partial p} = \frac{2p}{2m} \cdot \frac{1}{h} = \frac{p}{m} \cdot \frac{1}{h} = v \cdot \frac{1}{h}$$

See animation of group/phase velocity at:

http://en.wikipedia.org/wiki/Group_velocity

Resulting wave's "displacement" $y = y_1 + y_2$:

$$y = A\left[\cos(k_1 x - \omega_1 t) + \cos(k_2 x - \omega_2 t)\right]$$

Trignometry : $\cos A + \cos B = 2\cos(\frac{A+B}{2})\cos(\frac{A-B}{2})$

$k = \frac{2\pi}{\lambda}$

$\cos(kx - \omega t)$

$$\therefore y = 2A\left[\left(\cos(\frac{k_2 - k_1}{2}x - \frac{\omega_2 - \omega_1}{2}t)\right)\left(\cos(\frac{k_2 + k_1}{2}x - \frac{\omega_2 + \omega_1}{2}t)\right)\right]$$

since $k_2 \cong k_1 \cong k_{ave}$, $\omega_2 \cong \omega_1 \cong \omega_{ave}$, $\Delta k \ll k, \Delta\omega \ll \omega$

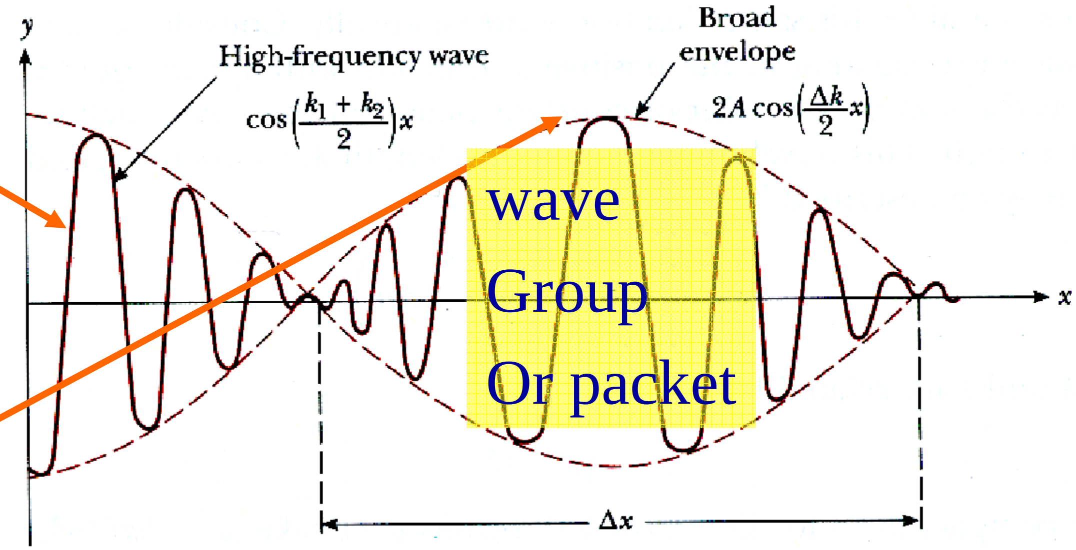
$$\therefore \boxed{y = 2A\left[\left(\cos(\frac{\Delta k}{2}x - \frac{\Delta\omega}{2}t)\right)\cos(kx - \omega t)\right]} \equiv y = A' \cos(kx - \omega t),$$ A' oscillates in x,t

$$A' = 2A\left(\cos(\frac{\Delta k}{2}x - \frac{\Delta\omega}{2}t)\right) = \text{modulated amplitude}$$

Phase Vel $V_p = \frac{w_{ave}}{k_{ave}}$

Group Vel $V_g = \frac{\Delta w}{\Delta k}$

V_g : Vel of envelope$= \frac{dw}{dk}$



Wave Packet : Localization

- Finite # of diff. Monochromatic waves always produce INFINTE sequence of repeating wave groups → can't describe (localized) particle
- To make localized wave packet, add " infinite" # of waves with Well chosen Ampl A, Wave# k, ang.

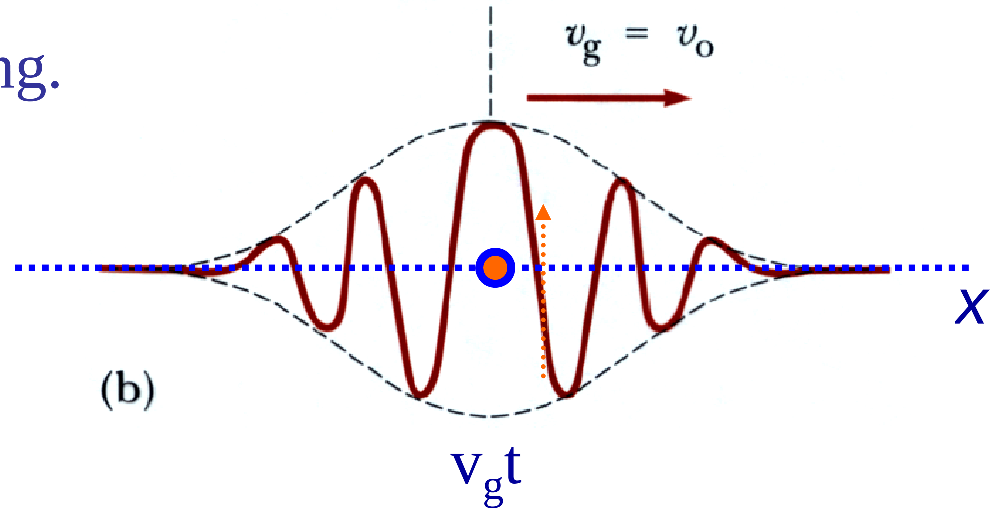
$$\psi(x,t) = \int_{-\infty}^{\infty} A(k)\, e^{i(kx-\omega t)} dk$$

$A(k)$ = Amplitude Fn

$\Rightarrow$ diff waves of diff k have different amplitudes A(k)

$\omega = \omega(k)$, depends on type of wave, media

Group Velocity $V_g = \left.\frac{d\omega}{dk}\right|_{k=k_0}$



Group, Velocity, Phase Velocity and Dispersion

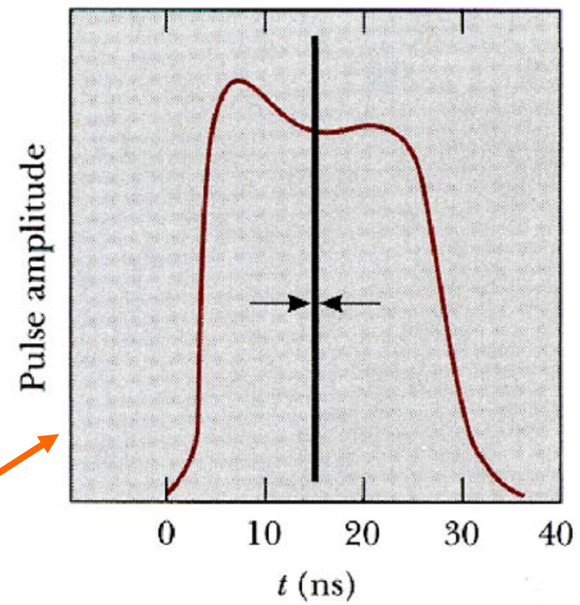
In a Wave Packet: $\omega = \omega(k)$

Group Velocity $V_g = \left.\frac{d\omega}{dk}\right|_{k=k_0}$

Since $V_p = \omega / k \quad (def) \Rightarrow \omega = kV_p$

$$\therefore \boxed{V_g = \left.\frac{d\omega}{dk}\right| = V_p\Big|_{k=k_0} + k\left.\frac{dV_p}{dk}\right|_{k=k_0}}$$

$\frac{\omega}{k} = c$



1ns laser pulse disperse By x30 after travelling 1km in optical fiber

usually $V_p = V_p(k \ or \lambda)$

Material in which V_p varies with λ are said to be Dispersive

Individual harmonic waves making a wave pulse travel at different V_p thus changing shape of pulse and become spread out

In non-dispersive media, $V_g = V_p$

In dispersive media $V_g \neq V_p$, depends on $\frac{dV_p}{dk}$

Matter Wave Packets

Consider An Electron:

mass = m velocity = v, momentum = p

Energy $E = hf = \gamma mc^2$; $\omega = 2\pi f = \frac{2\pi}{h}\gamma mc^2$

Wavelength $\lambda = \frac{h}{p}$; $k = \frac{2\pi}{\lambda} \Rightarrow k = \frac{2\pi}{h}\gamma mv$

Group Velocity : $V_g = \frac{d\omega}{dk} = \frac{d\omega / dv}{dk / dv}$

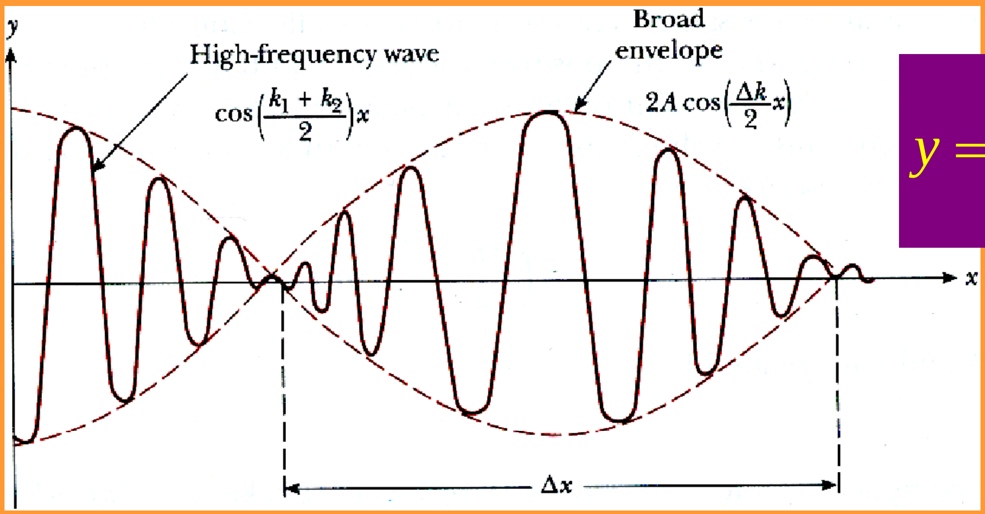
$$\frac{d\omega}{dv} = \frac{d}{dv}\left[\frac{\frac{2\pi}{h}mc^2}{[1-(\frac{v}{c})^2]^{1/2}}\right] = \frac{2\pi mv}{h[1-(\frac{v}{c})^2]^{3/2}} \quad \& \quad \frac{dk}{dv} = \frac{d}{dv}\left[\frac{2\pi}{h[1-(\frac{v}{c})^2]^{1/2}}mv\right] = \frac{2\pi m}{h[1-(\frac{v}{c})^2]^{3/2}}$$

$V_g = \frac{d\omega}{dk} = \frac{d\omega / dv}{dk / dv} = v \Rightarrow$ Group velocity of electron Wave packet "pilot wave"

is same as electron's physical velocity

But velocity of individual waves making up the wave packet $V_p = \frac{\omega}{k} = \frac{c^2}{v} > c$! (not physical)

Wave Packets & Uncertainty Principle



$$y = 2A\left[\left(\cos\left(\frac{\Delta k}{2}x - \frac{\Delta \omega}{2}t\right)\right)\cos(kx - \omega t)\right]$$

Amplitude Modulation

- Distance ΔX between adjacent minima = $(X_2)_{node}$ - $(X_1)_{node}$
- Define X_1=0 then phase diff from $X_1 \rightarrow X_2 = \pi$

Node at y = 0 = 2A cos $\left(\frac{\Delta\omega}{2}t - \frac{\Delta k}{2}x\right)$

$\Rightarrow$ $\boxed{\Delta k.\Delta x = 2\pi}$ $\Rightarrow$ Need to combine more k to make small Δx packet

also implies $\Rightarrow$ $\boxed{\Delta p.\Delta x = h}$

and

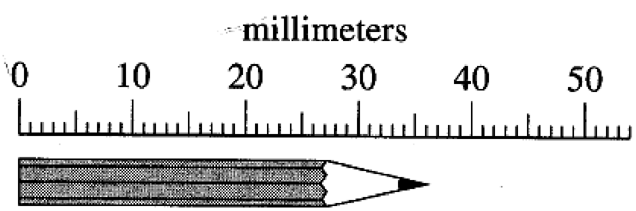
$\boxed{\Delta\omega.\Delta t = 2\pi}$ $\Rightarrow$ Need to combine more ω to make small Δt packet

also $\Rightarrow$ $\boxed{\Delta E.\Delta t = h}$

What does This mean?

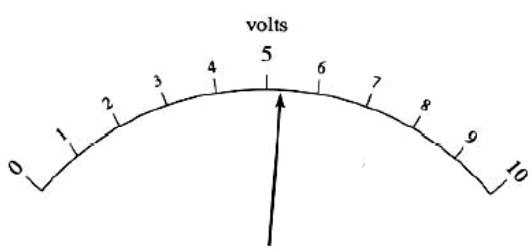
Know the Error of Thy Ways: Measurement Error → Δ

- Measurements are made by observing something : length, time, momentum, energy
- All measurements have some (limited) precision`…no matter the instrument used
- Examples:
  - How long is a desk ? L = (5 ± 0.1) m = L ± ΔL (depends on ruler used)
  - How long was this lecture ? T = (50 ± 1)minutes = T ± ΔT (depends on the accuracy of your watch)
  - How much does Prof. Sinha weigh ? M = (1000 ± 900) kg = m ± Δm
    - Is this a correct measure of my weight ?
      - Correct (because of large error reported) but imprecise
      - My correct weight is covered by the (large) error in observation



Best Estimate Length: 36 mm
Probable Range: 35.5 to 36.5 mm

Length Measure

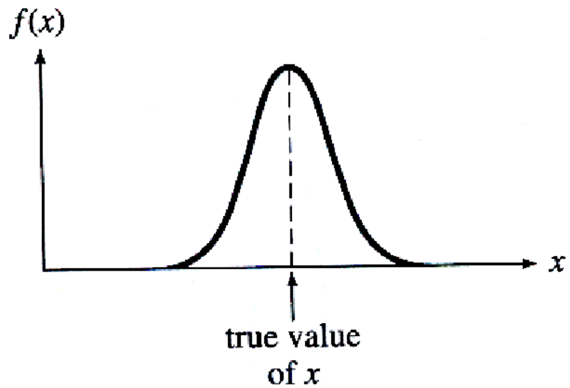


Best Estimate of Voltage: 5.3 V
Estimated Range: 5.2 to 5.4 mm

Voltage (or time) Measure

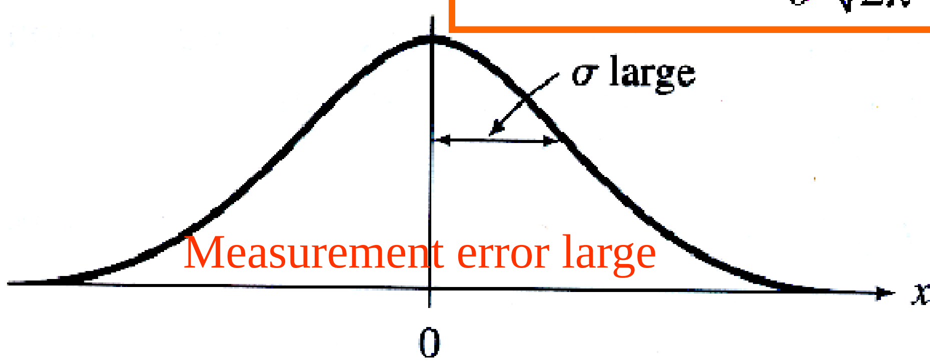
Measurement Error : x ± Δx

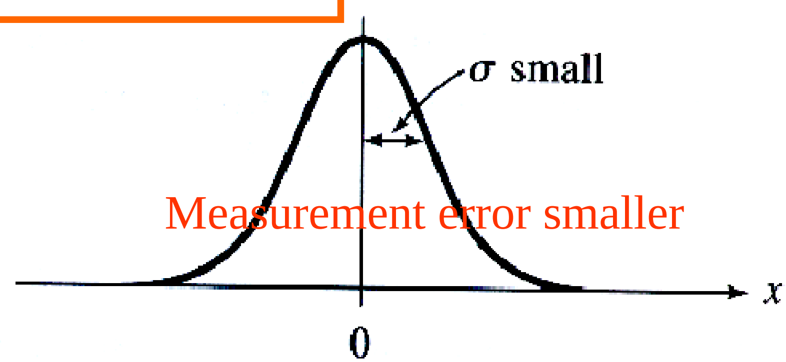
- Measurement errors are unavoidable since the measurement procedure is an experimental one
- True value of an measurable quantity is an abstract concept
- In a set of repeated <u>measurements with random errors</u>, the distribution of measurements resembles a Gaussian distribution characterized by the parameter σ or Δ characterizing the width of the distribution



The Gauss, or Normal, Distribution

$$G_{X,\sigma}(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{(x-X)^2/2\sigma^2}.$$





Measurement Error : x ± Δx

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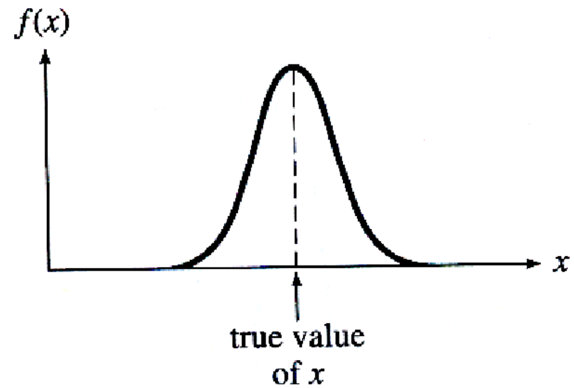
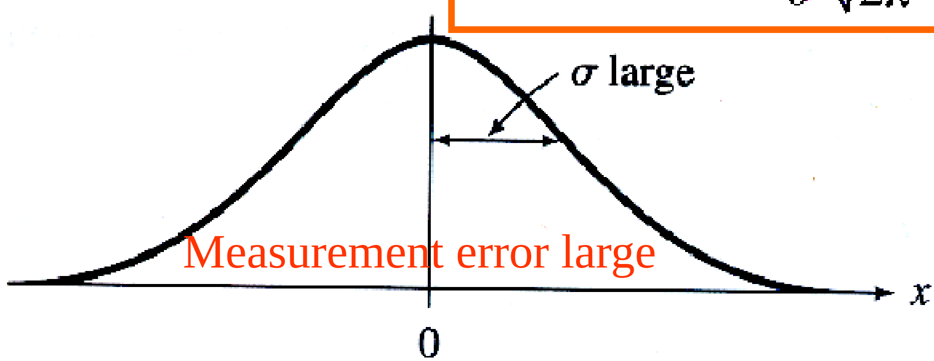
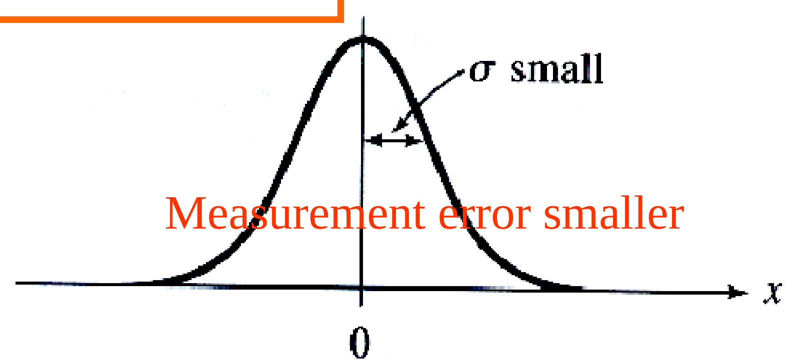


The Gauss, or Normal, Distribution

$$G_{X,\sigma}(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{(x - X)^2/2\sigma^2}.$$





Interpreting Measurements with random Error : Δ

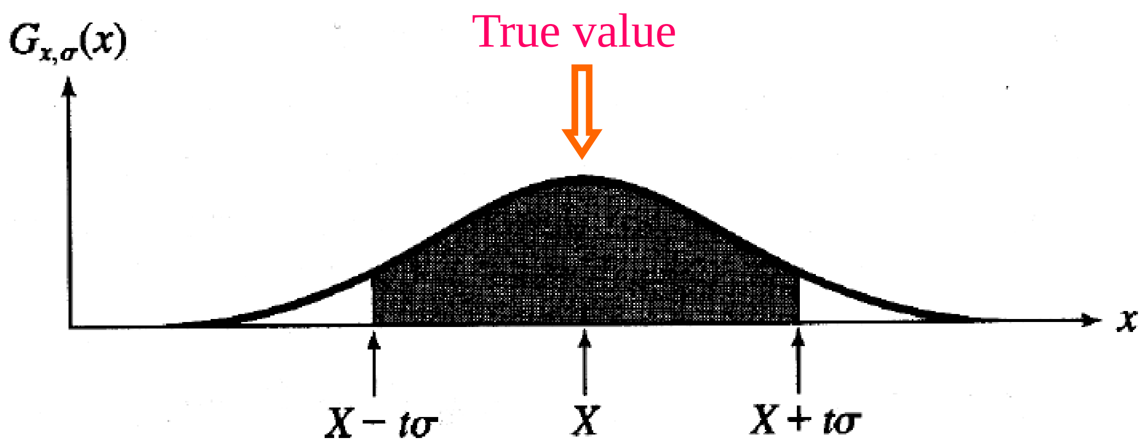
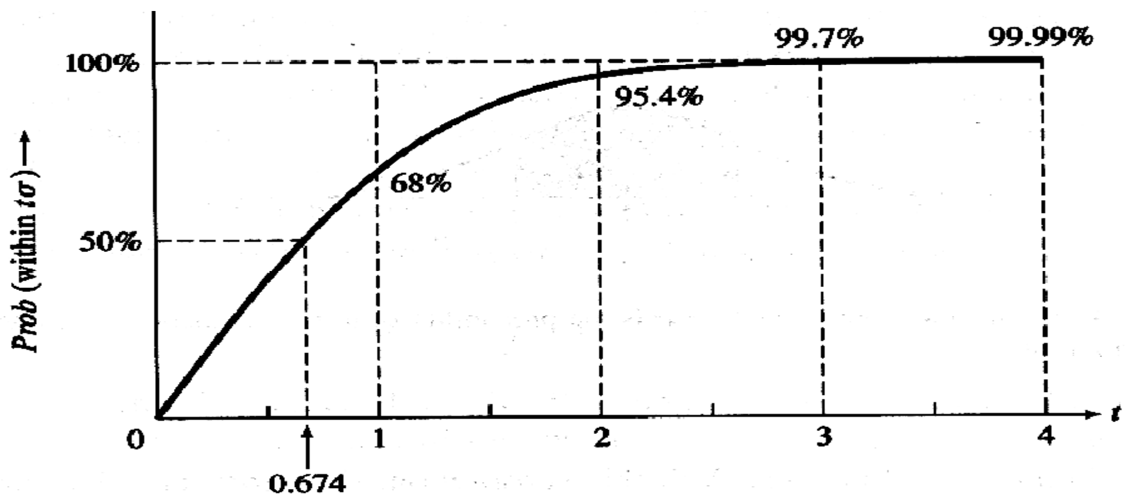


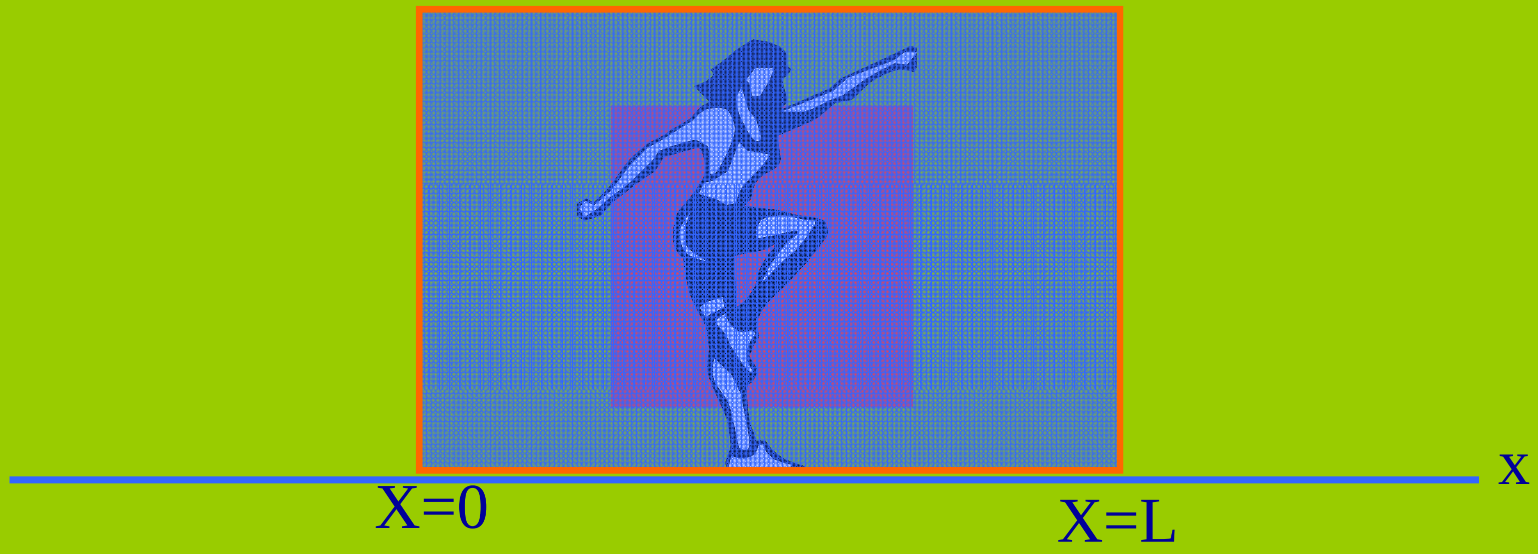
Figure 5.12. The shaded area between $X \pm t\sigma$ is the probability of a measurement within t standard deviations of X.



t	0	0.25	0.5	0.75	1.0	1.25	1.5	1.75	2.0	2.5	3.0	3.5	4.0
Prob (%)	0	20	38	55	68	79	87	92	95.4	98.8	99.7	99.95	99.99

Where in the World is Carmen San Diego?

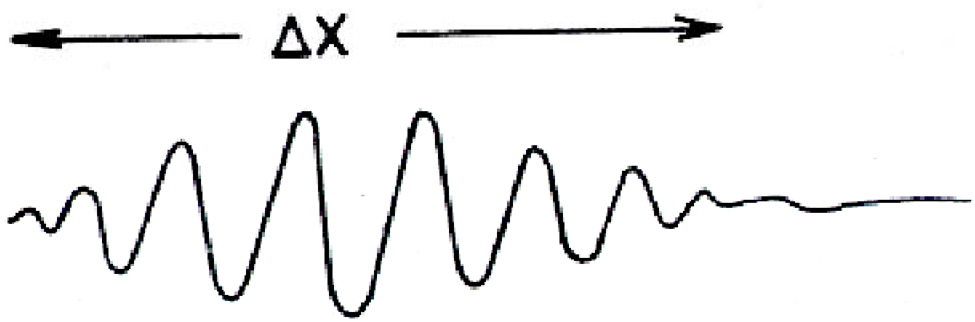
- Carmen San Diego hidden inside a big box of length L
- Suppose you can't see thru the (blue) box, what is you best estimate of her location inside box (she could be anywhere inside the box)



Your best unbiased measure would be $x = L/2 \pm L/2$

There is no perfect measurement, there are always measurement error

Wave Packets & Matter Waves



What is the Wave Length of this wave packet?

$\lambda - \Delta\lambda < \lambda < \lambda + \Delta\lambda$

De Broglie wavelength $\lambda = h/p$

→Momentum Uncertainty: $p - \Delta p < p < p + \Delta p$

Similarly for frequency ω or f

$\omega - \Delta\omega < \omega < \omega + \Delta\omega$

Planck's condition E= hf = hω/2

→ $E - \Delta E < E < E + \Delta E$

Back to Heisenberg’s Uncertainty Principle & Δ

- $\Delta x.\ \Delta p \geq h/4\pi \Rightarrow$
  - If the measurement of the position of a particle is made with a precision Δx and a SIMULTANEOUS measurement of its momentum p_x in the X direction , then the product of the two uncertainties (measurement errors) can never be smaller than $\cong h/4\pi$ irrespective of how precise the measurement tools

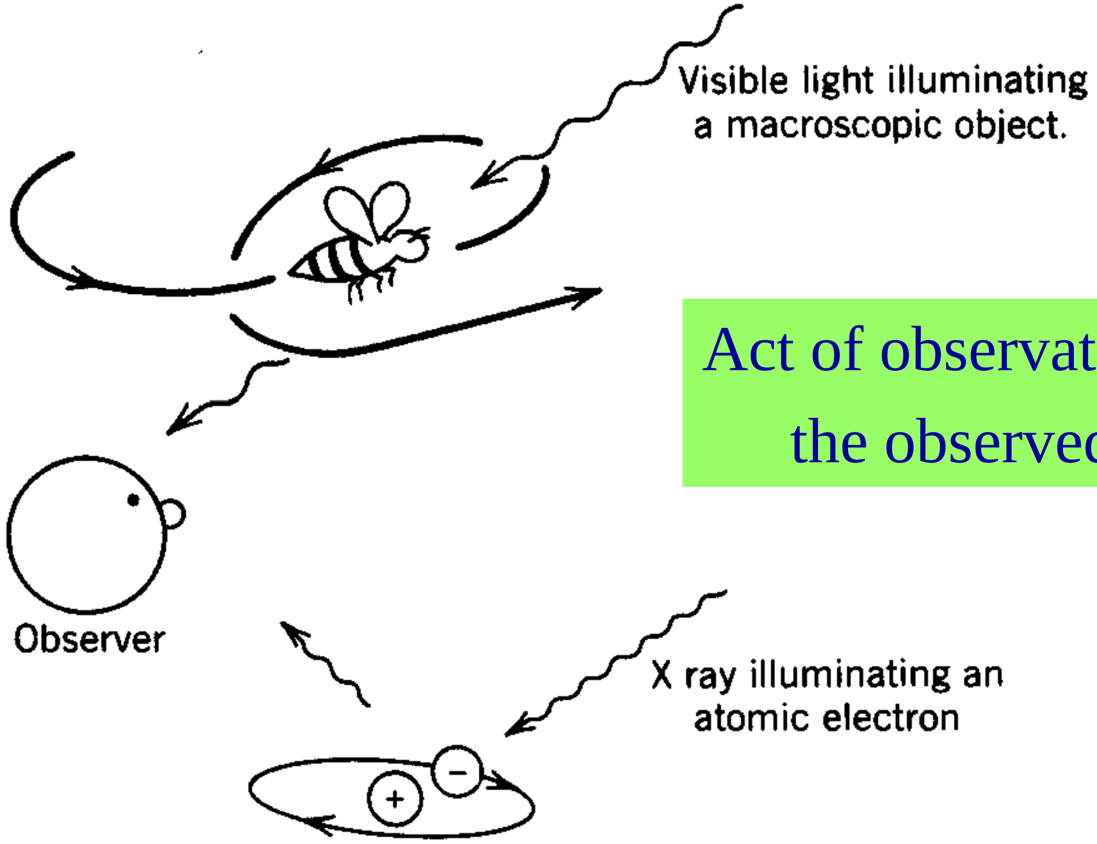
- $\Delta E.\ \Delta t \geq h/4\pi \Rightarrow$
  - If the measurement of the energy E of a particle is made with a precision ΔE and it took time Δt to make that measurement, then the product of the two uncertainties (measurement errors) can never be smaller than $\cong h/4\pi$ irrespective of how precise the measurement tools

These rules arise from the way we constructed the Wave packets describing Matter “pilot” waves

Perhaps these rules Are bogus, can we verify this with some physical picture ??

The Act of Observation (Compton Scattering)

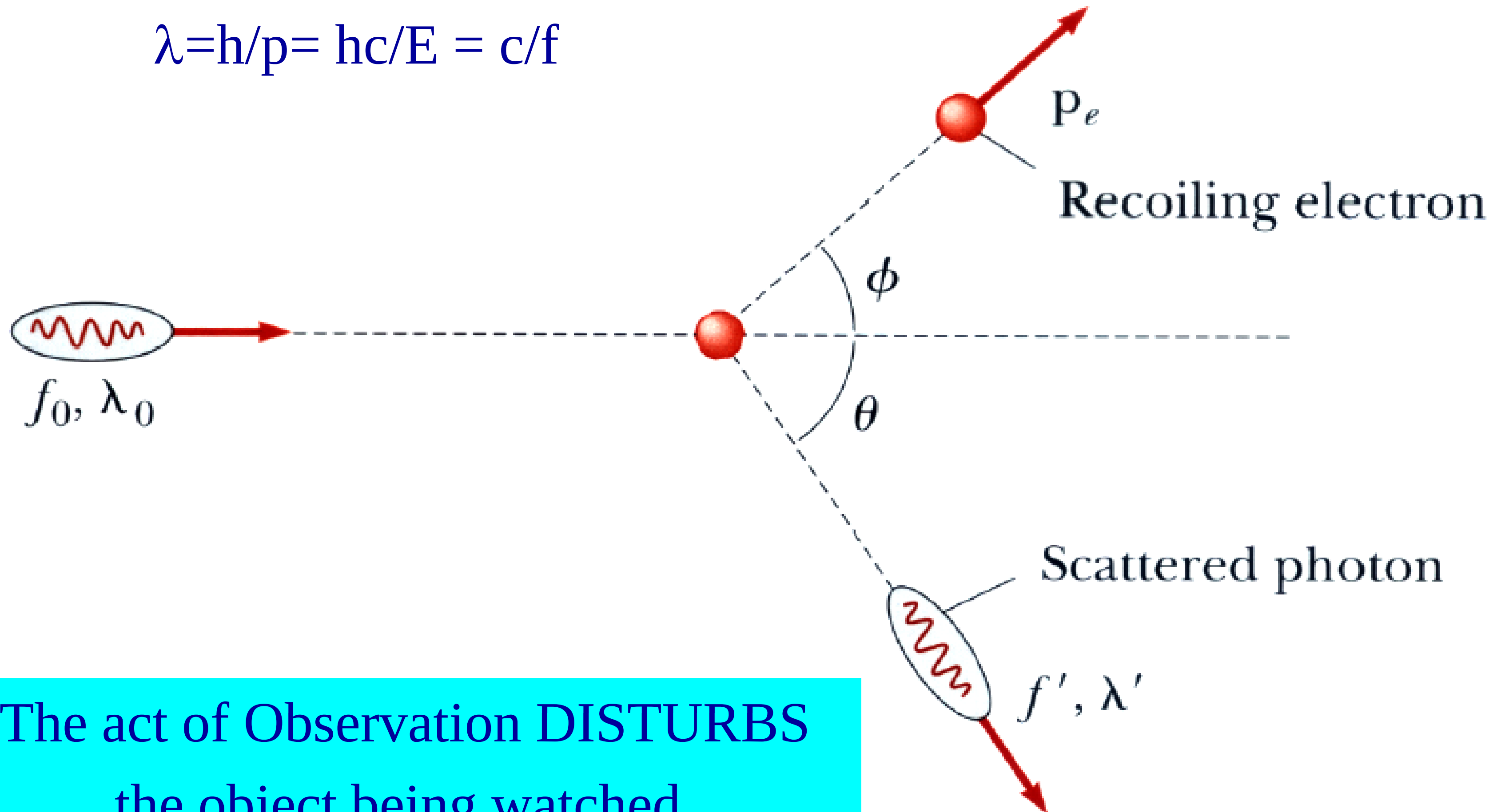
Observations of particle motion by means of scattered illumination. When the incident wavelength is reduced to accommodate the size of the particle, the momentum transferred by the photon becomes large enough to disturb the observed motion.



Act of observation disturbs the observed system

Compton Scattering: Shining light to observe electron

$\lambda = h/p = hc/E = c/f$



The act of Observation DISTURBS
the object being watched,
here the electron moves away from
where it was originally

Diffraction By a Circular Aperture (Lens)

See Resnick, Halliday Walker 6^{th} Ed (on S.Reserve), Ch 37, pages 898-900

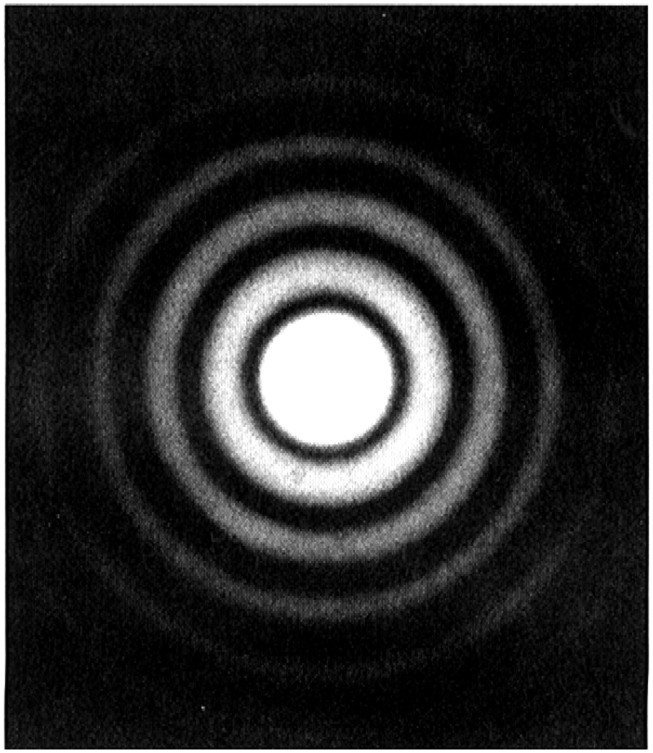


Fig. 37-9 The diffraction pattern of a circular aperture. Note the central maximum and the circular secondary maxima. The figure has been overexposed to bring out these secondary maxima, which are much less intense than the central maximum.

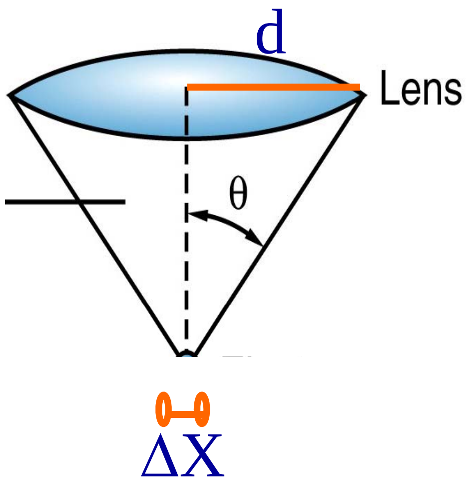
Diffracted image of a point source of light thru a lens (circular aperture of size d)

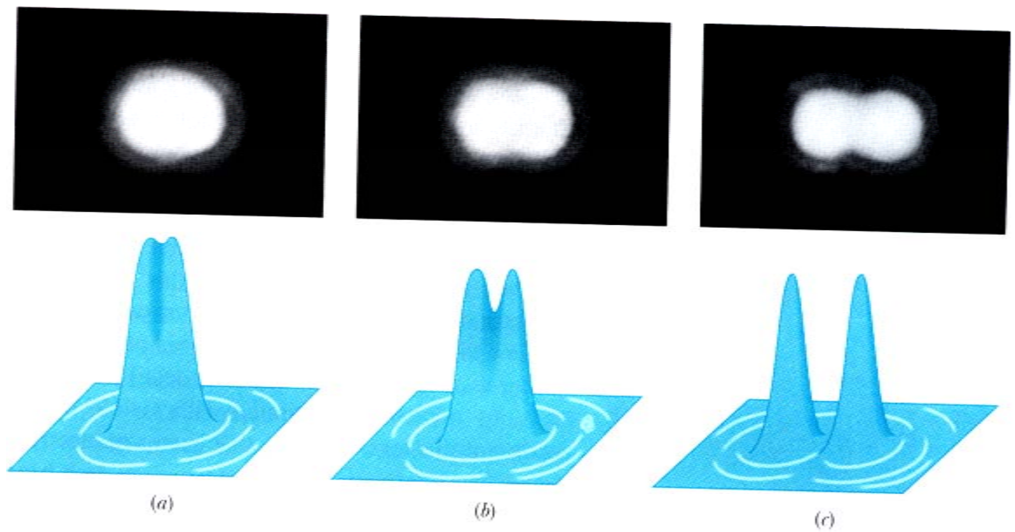
First minimum of diffraction pattern is located by

$$\sin\theta = 1.22\frac{\lambda}{d}$$

Resolving Power of Light Thru a Lens

Image of 2 separate point sources formed by a converging lens of diameter d, ability to resolve them depends on λ & d because of the Inherent diffraction in image formation

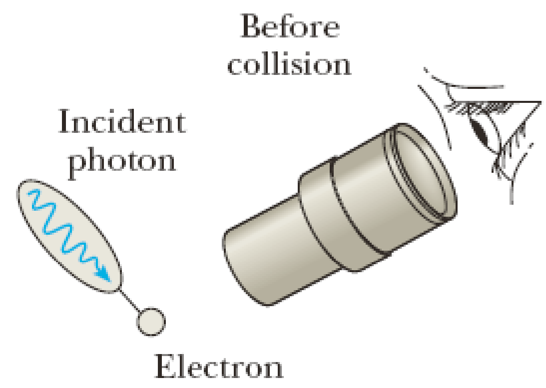




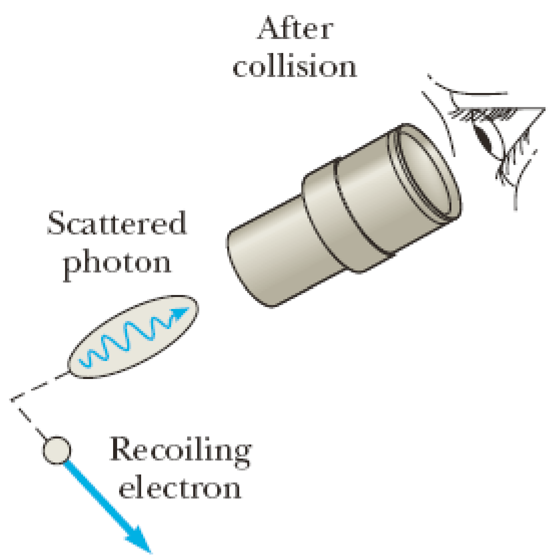
Not resolved barely resolved resolved

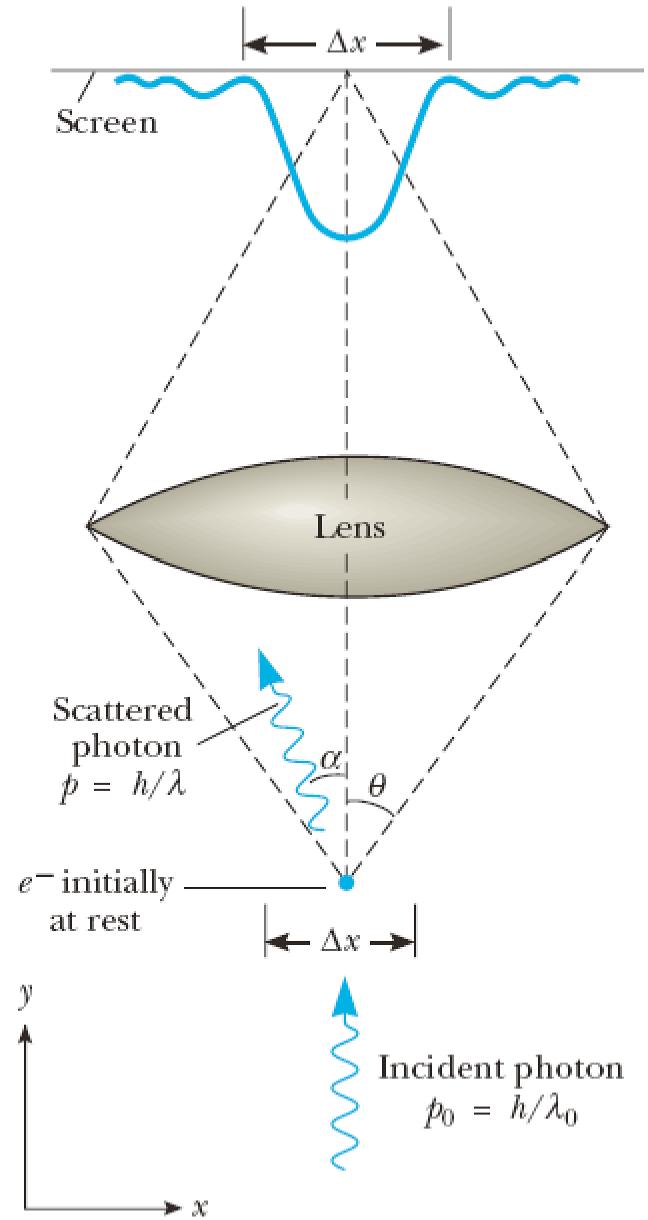
$$\text{Resolving power } \Delta x \approx \frac{\lambda}{2\sin\theta}$$

ϑ Depends on d

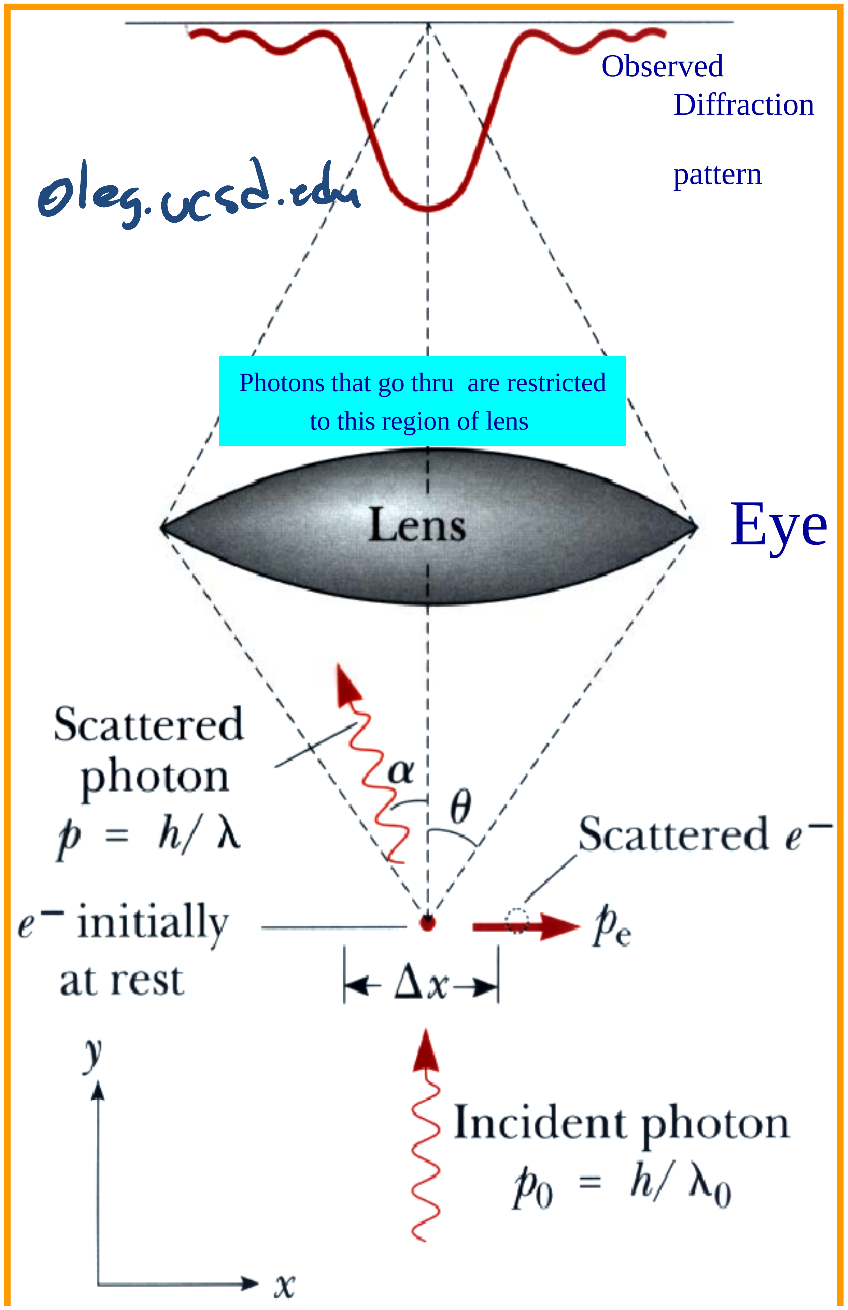

Before
collision
Incident
photon
Electron

(a)


After
collision
Scattered
photon
Recoiling
electron


Δx
Screen
Lens
Scattered
photon
$p = h/\lambda$
α
θ
e^- initially
at rest
Δx
y
x
Incident photon
$p_0 = h/\lambda_0$

Putting it all together: act of Observing an electron



- Incident light (p,λ) scatters off electron
- To be collected by lens → γ must scatter thru angle α
  - $-\vartheta \leq \alpha \leq \vartheta$
- Due to Compton scatter, electron picks up momentum
  - P_X , P_Y

$$-\frac{h}{\lambda}\sin\theta \leq P_x \leq \frac{h}{\lambda}\sin\theta$$

electron momentum uncertainty is

$$\Delta p \cong \frac{\sim 2h}{\lambda}\sin\theta$$

- After passing thru lens, photon diffracts, lands somewhere on screen, image (of electron) is fuzzy
- How fuzzy ? Optics says shortest distance between two resolvable points is :

$$\Delta x = \frac{\lambda}{2\sin\theta}$$

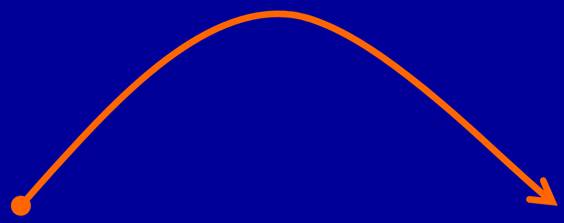
- Larger the lens radius, larger the $\vartheta \Rightarrow$ better resolution

$$\Rightarrow \Delta p.\Delta x \approx \left(\frac{2h\sin\theta}{\lambda}\right)\left(\frac{\lambda}{2\sin\theta}\right) = h$$

$$\Rightarrow \Delta p.\Delta x \geq \hbar/2$$

Pseudo-Philosophical Aftermath of Uncertainty Principle

- Newtonian Physics & Deterministic physics topples over
  - Newton's laws told you all you needed to know about trajectory of a particle
    - Apply a force, watch the particle go !



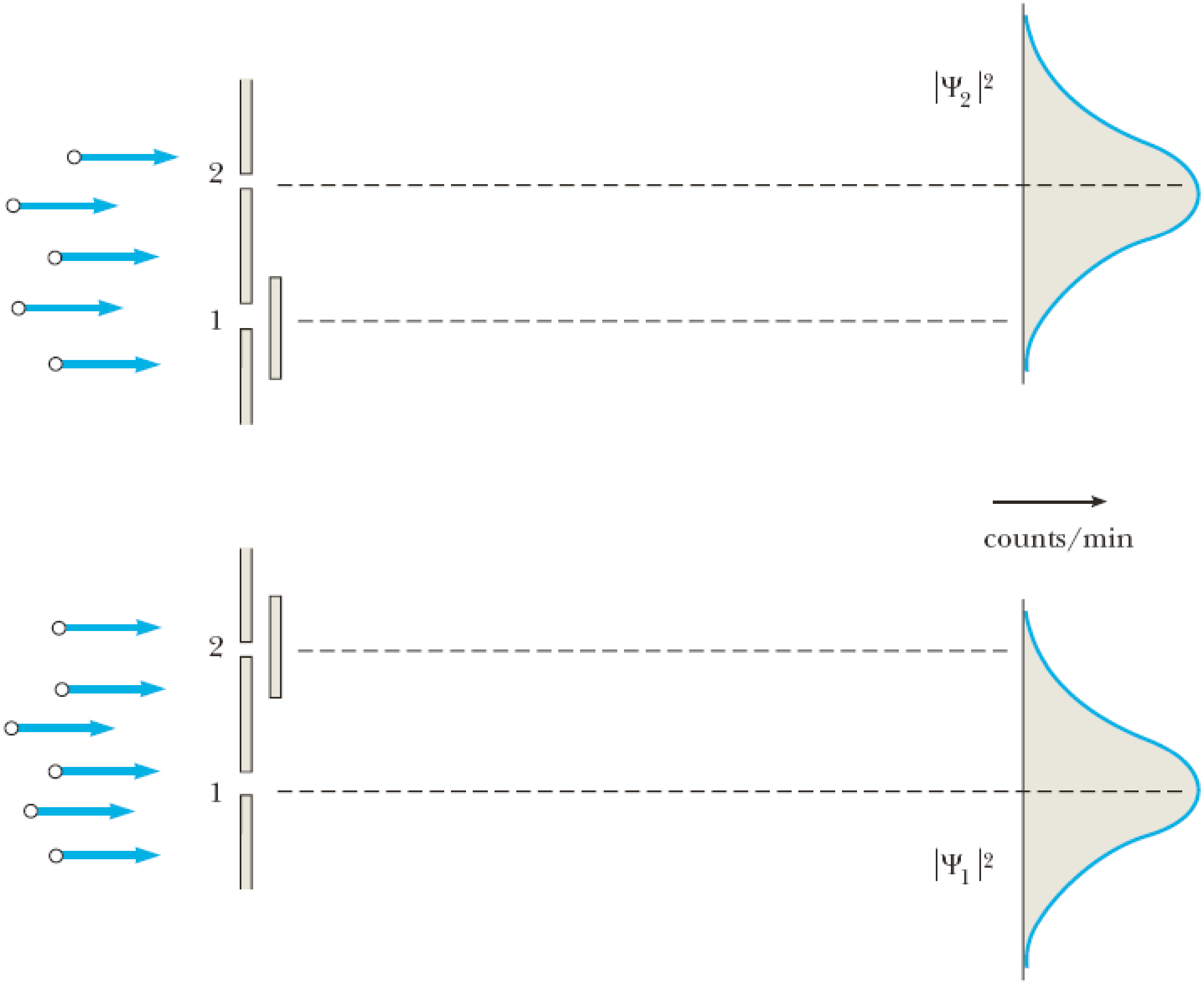
      - Know every thing ! X, v, p , F, a
      - Can predict exact trajectory of particle if you had perfect device
- No so in the subatomic world !
  - Of small momenta, forces, energies
  - Cant predict anything exactly
    - Can only predict probabilities
      - There is so much chance that the particle landed here or there
      - Cant be sure !....cognizant of the errors of thy observations

Philosophers went nuts !...what has happened to nature

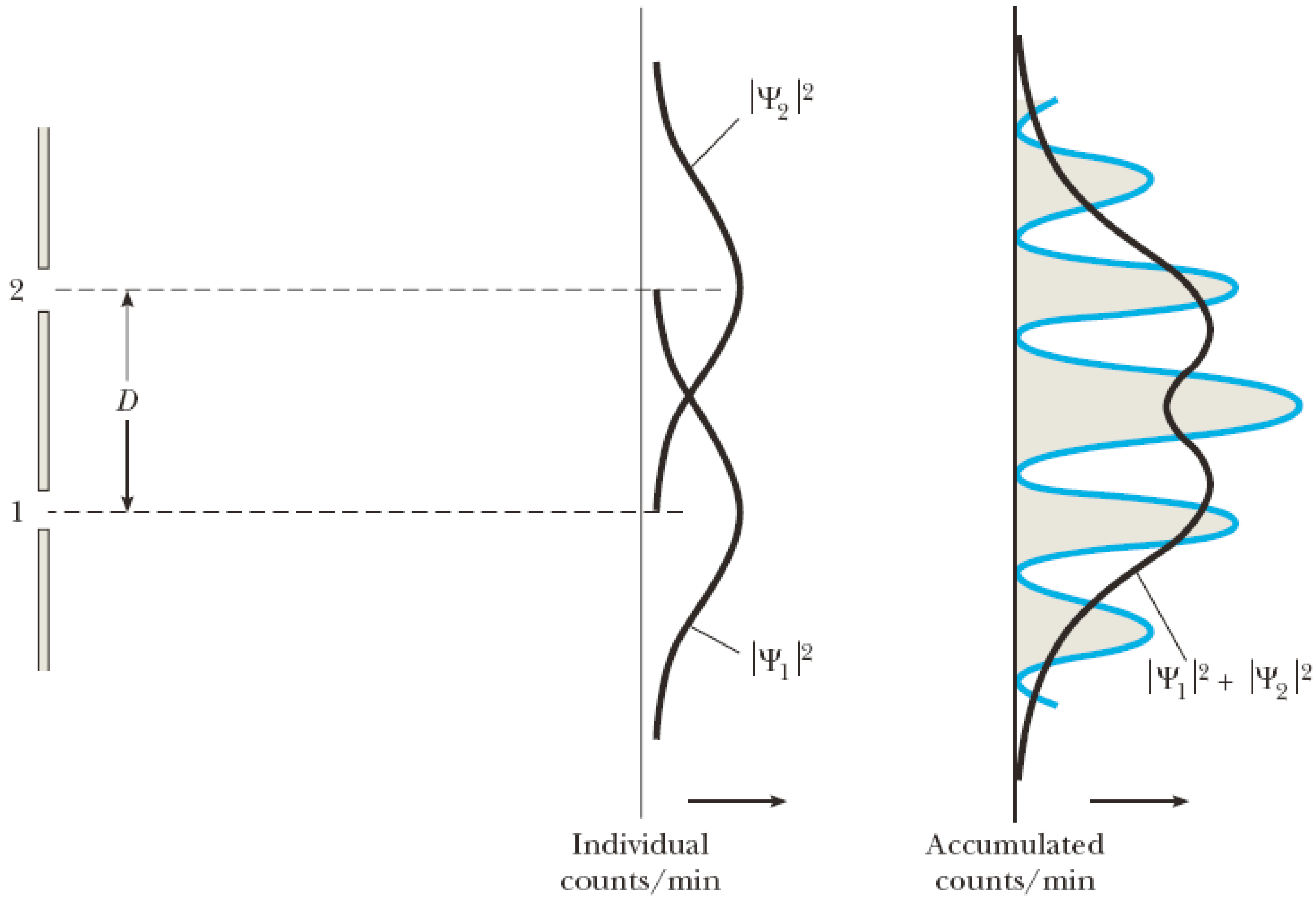
Philosophers just talk, don't do real life experiments!

All Measurements Have Associated Errors

- If your measuring apparatus has an intrinsic inaccuracy (error) of amount Δp
- Then results of measurement of momentum p of an object at rest can easily yield a range of values accommodated by the measurement imprecision :
  - $-\Delta p \leq p \leq \Delta p$
- Similarly for all measurable quantities like x, t, Energy !



$|\Psi_2|^2$
2
1
counts/min
2
1
$|\Psi_1|^2$



2
D
1
|Ψ2|2
|Ψ1|2
|Ψ1|2 + |Ψ2|2
Individual
counts/min
Accumulated
counts/min