PHYS 100C, Lecture 25

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Last lecture we learned how $\vec{E}$, $\vec{B}$ transform:

$$\tilde{E}_x = E_x$$
$$\tilde{E}_y = \gamma(E_y - vB_z)$$
$$\tilde{E}_z = \gamma(E_z + vB_y)$$
$$\tilde{B}_x = B_x$$
$$\tilde{B}_y = \gamma(B_y + \tfrac{v}{c^2}E_z)$$
$$\tilde{B}_z = \gamma(B_z - \tfrac{v}{c^2}E_y)$$

We would like to keep using Lorentz transform, for 4-vectors:

$$\tilde{a} = \Lambda \cdot a \qquad \text{OR:}$$

$$\tilde{a}^\mu = \sum_\nu \Lambda^\mu_{\ \nu} \cdot a^\nu$$

Often $\sum$ is dropped - summation over common index is implied (Einstein's notations)

$$\begin{pmatrix} \tilde{a}^0 \\ \tilde{a}^1 \\ \tilde{a}^2 \\ \tilde{a}^3 \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a^0 \\ a^1 \\ a^2 \\ a^3 \end{pmatrix}$$

$$\underbrace{\tilde{a}}_{\uparrow} \qquad \underbrace{\Lambda}_{\uparrow} \qquad \underbrace{a}_{\uparrow}$$

$\Lambda^{\mu}{}_{\nu}$ is element of μ row, ν column

We need to represent 6 numbers ($E_x, E_y, E_z, B_x, B_y, B_z$) - not possible with 4-vectors (4×1 matrix).

It's possible to do with 4×4 matrix, but it's too much "real estate" - 16 elements

$$\begin{pmatrix} t^{00} & t^{01} & t^{02} & t^{03} \\ t^{10} & t^{11} & t^{12} & t^{13} \\ t^{20} & t^{21} & t^{22} & t^{23} \\ t^{30} & t^{31} & t^{32} & t^{33} \end{pmatrix}$$

- 2^{nd} rank tensor, 16 elements

Symmetric tensor: $t^{\mu\nu} = t^{\nu\mu}$

Reduce 16 to $4 + 6 = 10$

independent variables

Antisymm. tensor: $t^{\mu\nu} = -t^{\nu\mu}$

implies: $t^{\mu\mu} = 0$

$$\begin{pmatrix} 0 & t^{01} & t^{02} & t^{03} \\ -t^{01} & 0 & t^{12} & t^{13} \\ -t^{02} & -t^{12} & 0 & t^{23} \\ -t^{03} & -t^{13} & -t^{23} & 0 \end{pmatrix}$$

ONLY 6 elem. independent

We will need to use Λ multiplication twice - once for each index:

$$\tilde{t}^{\mu\nu} = \sum_{\lambda,\sigma} \Lambda^{\mu}_{\lambda} \Lambda^{\nu}_{\sigma} \cdot t^{\lambda\sigma}$$

Let's consider $\tilde{t}^{01} = \sum_{\lambda,\sigma} \Lambda^{0}_{\lambda} \Lambda^{1}_{\sigma} t^{\lambda\sigma}$

but $\Lambda^{0}_{\lambda} = 0$ FOR $\lambda = 2, 3$

$\Lambda^{1}_{\sigma} = 0$ FOR $\sigma = 2, 3$

$$\tilde{t}^{01} = \sum_{\substack{\lambda=0,1 \\ \sigma=0,1}} \Lambda^{0}_{\lambda} \Lambda^{1}_{\sigma} t^{\lambda\sigma} = \Lambda^{0}_{0} \Lambda^{1}_{0} t^{00} + \Lambda^{0}_{0} \Lambda^{1}_{1} t^{01} + \Lambda^{0}_{1} \Lambda^{1}_{0} t^{10} + \Lambda^{0}_{1} \Lambda^{1}_{1} t^{11}$$

but since $t^{00} = t^{11} = 0$ (antisym.)

$$\tilde{t}^{01} = \gamma^2 t^{01} + \gamma^2 \beta^2 t^{10}$$

$$\tilde{f}^{01} = \gamma^2 f^{01} + \gamma^2\beta^2 f^{10}$$

$$f^{10} = -f^{01} \Rightarrow$$

$$\tilde{f}^{01} = (\gamma^2 - \gamma^2\beta^2) f^{01} = \gamma^2 \underbrace{(1-\beta^2)}_{1/\gamma^2} f^{01} =$$

$$= f^{01}$$

Can do the same calcutations for all $f^{\mu\nu}$, and find that f's transform the way E/c & B transform.

Can introduce 2nd rank, antisymm. tensors:

$$F^{\mu\nu} = \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{pmatrix}$$

OR, ALTERNATIVELY:

$$G^{\mu\nu} = \begin{pmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & -E_z/c & E_y/c \\ -B_y & E_z/c & 0 & -E_x/c \\ -B_z & -E_y/c & \frac{E_x}{c} & 0 \end{pmatrix}$$

Then:

$$\tilde{F} = \Lambda \cdot \Lambda \cdot F$$

$$\left(\text{OR} \quad \tilde{F}^{\mu\nu} = \sum_{\lambda,\sigma} \Lambda^{\mu}_{\ \lambda} \Lambda^{\nu}_{\ \sigma} \cdot F^{\lambda\sigma} \right)$$

and

$$\tilde{G} = \Lambda \cdot \Lambda \cdot G$$

* AND NOW, FOR something completely different:

charge density: $\rho = \frac{Q}{V}$

current density: $J = \rho \cdot u$

In moving frame $V = \frac{V_0}{\gamma}$

due to Lorentz contraction along the direction of motion.

$$\tilde{\rho} = \rho_0 \cdot \gamma$$

$$\tilde{J} = \rho_0 \cdot u \cdot \gamma$$

IF we introduce $\rho \cdot c$,

then it looks just like $\rho_0 \cdot u^0$

(proper velocity), and

"$\tilde{J} = \rho_0 \cdot \vec{U}^\mu$"

4-vector: $J^\mu = \rho_0 \cdot u^\mu = \rho_0 \begin{pmatrix} c \\ u_x \\ u_y \\ u_z \end{pmatrix}$

OR $J^\mu = \begin{pmatrix} \rho c \\ J_x \\ J_y \\ J_z \end{pmatrix}$ transforms like any other 4-vector.

Conservation of charge
(absence of "FREE LUNCH" CONCEPT)

$$\nabla \cdot J = -\frac{\partial \rho}{\partial t}$$

$$\frac{\partial J_x}{\partial x} + \frac{\partial J_y}{\partial y} + \frac{\partial J_z}{\partial z} = -\frac{\partial \rho}{\partial t}$$

$$\frac{\partial \rho}{\partial t} = \frac{\partial \overbrace{c \cdot \rho}^{J^0}}{\partial \underbrace{(c \cdot t)}_{x^0}} = \frac{\partial J^0}{\partial x^0}$$

$$\frac{\partial J^0}{\partial x^0} + \frac{\partial J^1}{\partial x^1} + \frac{\partial J^2}{\partial x^2} + \frac{\partial J^3}{\partial x^3} = 0$$

OR $$\frac{\partial J^\mu}{\partial x^\mu} = 0$$

(summation over $\mu = 0, 1, 2, 3$ is implied)!