PHYS 100C Midterm, Thursday May 10, 12:30-1:50 (1hr 20min)

1. Write down the general expression for (real component) of electric field (magnitude E_0) and magnetic field for plane monochromatic wave of frequency ω propagating in negative direction along y-axis and polarization along x-axis, so that t=0 and y=0 corresponds to a node E=B=0.
2. In deep water waves travel at speed that is proportional to square root of wavelength, λ.
   (a) Find a ratio between phase and group velocities for these types of waves.
   (b) What is the answer if speed of waves was instead proportional to an arbitrary power of λ^N?
3. (a) Show that the solutions for coaxial transmission line (in cylindrical coordinates), Eq. 9.197:

$$\left.\begin{aligned} \mathbf{E}(s,\phi,z,t) &= \frac{A\cos(kz-\omega t)}{s}\,\hat{\mathbf{s}}, \\ \mathbf{B}(s,\phi,z,t) &= \frac{A\cos(kz-\omega t)}{cs}\,\hat{\boldsymbol{\phi}}. \end{aligned}\right\}$$

   satisfy the Maxwell Equations

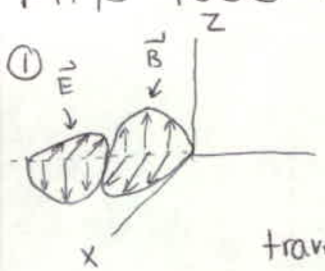
$$\left.\begin{aligned} &\text{(i)}\ \nabla\cdot\mathbf{E}=0, \quad &&\text{(iii)}\ \nabla\times\mathbf{E}=-\frac{\partial\mathbf{B}}{\partial t}, \\ &\text{(ii)}\ \nabla\cdot\mathbf{B}=0, \quad &&\text{(iv)}\ \nabla\times\mathbf{B}=\frac{1}{c^2}\frac{\partial\mathbf{E}}{\partial t}. \end{aligned}\right\}$$

   and boundary conditions (9.175)

$$\left.\begin{aligned} &\text{(i)} \quad \mathbf{E}^{\parallel}=0, \\ &\text{(ii)} \quad B^{\perp}=0. \end{aligned}\right\}$$

   (b) Find the charge density, $\lambda(z,t)$, and the current, $I(z,t)$ on the inner conductor.

PHYS 100C Midterm Solutions

① 

$\hat{k} = -\hat{y}$

$\hat{n} = \hat{x}$ (x-polarized)

$(-\hat{y}) \times (\hat{x}) = \hat{z}$ ($\vec{B}$-field direction)

travels in $-y$-direction $\quad \vec{k}\cdot\vec{r} = \frac{\omega}{c} y = ky$

node $E = B = 0$ at $y = t = 0 \rightarrow$ use sines not cosines!

$$\boxed{\begin{aligned}\vec{E} &= E_0 \sin\left(\frac{\omega}{c} y + \omega t\right)\hat{x} \\ \vec{B} &= \frac{E_0}{c}\sin\left(\frac{\omega}{c} y + \omega t\right)\hat{z}\end{aligned}}$$

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② when you are watching a water wave travel, you are looking at a single component $\rightarrow$ phase velocity!

a) $v_p \propto \lambda^{1/2}$ where $v_p = \frac{\omega}{k}$ and $v_g = \frac{d\omega}{dk}$ with $k = \frac{2\pi}{\lambda}$

$$v_p = \frac{\omega}{k} = \underset{\text{constant}}{\underset{\uparrow}{A}}\lambda^{1/2}$$

$$k = \frac{2\pi}{\lambda} \rightarrow \lambda = \frac{2\pi}{k} \rightarrow \lambda^{1/2} = \frac{\sqrt{2\pi}}{k^{1/2}}$$

$$\frac{\omega}{k} = A\frac{\sqrt{2\pi}}{k^{1/2}}$$

$$\omega = A\sqrt{2\pi k}$$

$$\frac{d\omega}{dk} = \frac{A}{2\sqrt{2\pi k}}\cdot 2\pi = \frac{A\sqrt{2\pi}}{2k^{1/2}}$$

$$\frac{v_p}{v_g} = \frac{\omega/k}{d\omega/dk} = \frac{\cancel{A}\frac{\cancel{\sqrt{2\pi}}}{\cancel{k^{1/2}}}}{\cancel{A}\frac{\cancel{\sqrt{2\pi}}}{2\cancel{k^{1/2}}}} = \boxed{2}$$

b) $v_p \propto \lambda^N$

$$v_p = \frac{\omega}{k} = A\lambda^N = A\left(\frac{2\pi}{k}\right)^N$$

$$\frac{\omega}{k} = A\frac{(2\pi)^N}{k^N}$$

$$\omega = A\frac{(2\pi)^N}{k^{N-1}}$$

$$\frac{d\omega}{dk} = A(2\pi)^N \cdot -(N-1)\frac{1}{k^N}$$

$$\frac{v_p}{v_g} = \frac{\omega/k}{d\omega/dk} = \frac{\cancel{A}\frac{\cancel{(2\pi)^N}}{\cancel{k^N}}}{\cancel{A}\cancel{(2\pi)^N}\cdot\frac{1-N}{\cancel{k^N}}} = \boxed{\frac{1}{1-N}}$$

check for part a: $N = \frac{1}{2}$

$$\frac{v_p}{v_g} = \frac{1}{1-\frac{1}{2}} = \frac{1}{\frac{1}{2}} = 2$$

which is consistent!

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③ $\vec{E} = \frac{A\cos(kz-\omega t)}{s}\hat{s} \qquad \vec{B} = \frac{A\cos(kz-\omega t)}{cs}\hat{\phi}$

only need pieces from divergence and curl formulas which use E_s, B_ϕ

$$\nabla\cdot\vec{E} = \frac{1}{s}\frac{\partial}{\partial s}\left[\cancel{s}\frac{A\cos(kz-\omega t)}{\cancel{s}}\right] = 0 \checkmark$$

$$\nabla\times\vec{E} = \left[\frac{\partial}{\partial z}\frac{A\cos(kz-\omega t)}{s}\right]\hat{\phi} + \frac{1}{s}\left[-\frac{\partial}{\partial\phi}\cancel{\frac{A\cos(kz-\omega t)}{s}}\right]\hat{z} = \frac{-kA\sin(kz-\omega t)}{s}\hat{\phi}$$

$$-\frac{\partial\vec{B}}{\partial t} = -\frac{\partial}{\partial t}\left[\frac{A\cos(kz-\omega t)}{cs}\hat{\phi}\right] = \frac{-\omega A\sin(kz-\omega t)}{cs}\hat{\phi} = \frac{-kA\sin(kz-\omega t)}{s}\hat{\phi} \checkmark$$

since $\frac{\omega}{c} = k$

$$\nabla \cdot \vec{B} = \frac{1}{s}\frac{\partial}{\partial \phi} \frac{A\cos(kz-\omega t)}{cs} = 0 \checkmark$$

$$\nabla \times \vec{B} = -\left[\frac{\partial}{\partial z} \frac{A\cos(kz-\omega t)}{cs}\right]\hat{s} + \frac{1}{s}\left[\frac{\partial}{\partial s}\cancel{\frac{\cancel{s}\,A\cos(kz-\omega t)}{c\cancel{s}}}\right]\hat{z} = \frac{+kA\sin(kz-\omega t)}{cs}\,\hat{s}$$

$$\frac{1}{c^2}\frac{\partial \vec{E}}{\partial t} = \frac{1}{c^2}\cdot\frac{\partial}{\partial t}\left[\frac{A\cos(kz-\omega t)}{s}\right]\hat{s} = \frac{\omega A\sin(kz-\omega t)}{c^2 s}\hat{s} = \frac{kA\sin(kz-\omega t)}{cs}\,\hat{s} \checkmark$$

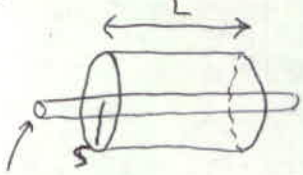
boundary conditions

(i) $\vec{E}^{\parallel} = 0$ note this is a vector! There are two directions parallel to the boundary surface! In cylindrical coordinates, these are:

$\boxed{E_z = E_\phi = 0}$ (since only E_s is nonzero)

(ii) $B^{\perp} = 0$ $\boxed{B_s = 0}$

b) use Gauss's Law $\nabla \cdot \vec{E} = \frac{1}{\epsilon_0}\rho$ or $\int \frac{1}{\epsilon_0}\rho\, dV = \oint \vec{E}\cdot d\vec{a}$



inner conductor

$\vec{E}$ is entirely radial → only nonzero on curved portion of Gaussian surface, of area $2\pi s L$

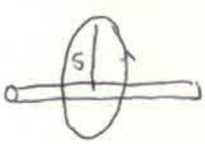
$\int \rho\, dV = Q$ (charge enclosed)

so $\frac{Q}{\epsilon_0} = E \cdot 2\pi s L$

charge per unit length $\lambda = \frac{Q}{L}$, so $\lambda = 2\pi s \epsilon_0 E$

$$\lambda = 2\pi \cancel{s} \epsilon_0 \frac{A\cos(kz-\omega t)}{\cancel{s}} \Rightarrow \boxed{\lambda = 2\pi\epsilon_0 A\cos(kz-\omega t)}$$

use Ampere's Law $\nabla\times\vec{B} = \mu_0\vec{J} + \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t}$ or $\int\left(\mu_0\vec{J} + \frac{1}{c^2}\frac{\partial\vec{E}}{\partial t}\right)\cdot d\vec{a} = \oint \vec{B}\cdot d\vec{l}$



$\frac{\partial \vec{E}}{\partial t}$ is in the $\hat{s}$ direction, while $d\vec{a}$ is in $\hat{z}$

so $\frac{1}{c^2}\frac{\partial\vec{E}}{\partial t}\cdot d\vec{a} = 0$

$$\int \mu_0 \vec{J}\cdot d\vec{a} = \oint \vec{B}\cdot d\vec{l}$$

$$B \cdot 2\pi s = \mu_0 I$$

$$I = \frac{B\cdot 2\pi s}{\mu_0} = \frac{A\cdot 2\pi \cancel{s}\cos(kz-\omega t)}{\mu_0 c \cancel{s}} \Rightarrow \boxed{I = \frac{2\pi A\cos(kz-\omega t)}{\mu_0 c}}$$