

# PHYSICS 100C, LECTURE # 7

Monday, April 12, 2010

10:01 AM

## EM Waves in Conductors

$$\vec{J} = \sigma \vec{E} \quad (\text{Ohm's Law})$$


$\vec{J}$  is current density

$\sigma$  is conductivity

\* Sanity check: traditional form of Ohm's Law  $I \cdot R = V$

$$I = J \cdot S \quad (S \text{ is area})$$

$$V = E \cdot \ell \quad (\ell \text{ is length})$$


$$R = \rho \frac{\ell}{S} = \frac{1}{\sigma} \cdot \frac{\ell}{S}$$
$$J \cdot S \cdot \frac{1}{\sigma} \cdot \frac{\ell}{S} = E \cdot \ell$$

OR:

$$J = \sigma E$$

Maxwell's Eqs, linear media, with free charges and currents:

$$\vec{\nabla} \cdot \vec{E} = \frac{1}{\epsilon} \rho \quad (\text{i})$$

$$\vec{\nabla} \cdot \vec{B} = 0 \quad (\text{ii})$$

$$\vec{\nabla} \times \vec{E} = - \frac{\partial \vec{B}}{\partial t} \quad (\text{iii})$$

$$\vec{\nabla} \times \vec{B} = \mu \sigma \vec{E} + \mu \epsilon \frac{\partial \vec{E}}{\partial t} \quad (\text{iv})$$

(we substituted  $\vec{J} = \sigma \vec{E}$  in (iv))

Conservation of charge  $\Rightarrow$   
similar to Gauss' law, flux  
of current outside closed volume

is equal to  $-\frac{\partial \rho}{\partial t}$ :


$$\vec{\nabla} \cdot \vec{J} = -\frac{\partial \rho}{\partial t}, \text{ or}$$

$$\frac{\partial \rho}{\partial t} = -\vec{\nabla} \cdot \vec{J} = -\sigma \cdot (\vec{\nabla} \cdot \vec{E}) = -\frac{\sigma}{\epsilon} \rho$$

Ohm's Law ↑ Maxwell (i)

Solution:  $\rho = \rho_0 \cdot \exp\left(-\frac{\sigma}{\epsilon} t\right)$

Free charge in conducting media "dissipates" (runs to the edges / leaks away) exponentially with characteristic time constant  $\tau = \frac{\epsilon}{\sigma}$



$\Rightarrow Q = Q_0 \cdot e^{-t/RC}$   
Similar to RC circuit

except media is both R and C

$$R \sim \frac{1}{\sigma} \quad C \sim \epsilon$$

"good" conductor  $\tau \equiv \frac{\epsilon}{\sigma} \ll \frac{1}{\omega}$

"bad" conductor  $\tau \gg \frac{1}{\omega}$

"perfect" conductor  $\sigma \rightarrow \infty$   
 $\tau \rightarrow 0$

If we wait  $\gg \tau$  for free charge to "dissipate",

$$\rho \rightarrow 0 \text{ in Eq. (i).}$$

Just like in Lecture #1  
Curl (iii) and (iv) gives:

$$\nabla^2 \vec{E} = \mu\epsilon \frac{\partial^2 \vec{E}}{\partial z^2} + \mu\sigma \frac{\partial \vec{E}}{\partial t}$$

$$\nabla^2 \vec{B} = \mu\epsilon \frac{\partial^2 \vec{B}}{\partial z^2} + \mu\sigma \frac{\partial \vec{B}}{\partial t}$$

this is new!

consequence of allowing  $\vec{j} \neq 0$   
(free currents)

General Solution is still a planar wave: (along z dir.)

$$\vec{E}(z,t) = \vec{E}_0 e^{i(\vec{k}z - \omega t)}$$

$$\vec{B}(z,t) = \vec{B}_0 e^{i(\vec{k}z - \omega t)}$$

just like for non-conduct. media, but now  $\vec{k}$  is a complex number:

$$\nabla^2 \vec{E} = -\vec{k}^2 \cdot \vec{E}$$

$$\frac{\partial \vec{E}}{\partial t} = -i\omega \vec{E} \quad (\text{same for } \vec{B})$$

$$\frac{\partial^2 \vec{E}}{\partial t^2} = -\omega^2 \vec{E}$$

new "wave eqs" above become:

$$\vec{k}^2 = \mu\epsilon \omega^2 + i\mu\sigma\omega \quad \text{new part}$$

Without currents there would be no  $i\mu\sigma\omega$  part (think  $\sigma \rightarrow 0$ , insulator) and  $\vec{k}$  is real  $\frac{\omega}{k} = v_1 = \frac{1}{\sqrt{\mu\epsilon}}$

Now  $\vec{k}$  is complex:

$$\vec{k} = \underbrace{k}_{\text{real}} + i\underbrace{\gamma}_{\text{IMAGINARY}} \quad \text{OR}$$

$$\vec{k}^2 = k^2 + 2ik\gamma - \gamma^2 = \mu\epsilon\omega^2 + i\mu\sigma\omega$$

$$\left. \begin{aligned} k^2 - \gamma^2 &= \mu\epsilon\omega^2 \\ 2k\gamma &= \mu\sigma\omega \end{aligned} \right\} \text{solve for } k, \gamma:$$

$$k = \omega \sqrt{\frac{\epsilon\mu}{2}} \left[ \sqrt{1 + \left(\frac{\sigma}{\epsilon\omega}\right)^2} + 1 \right]^{1/2}$$

$$\gamma = \omega \sqrt{\frac{\epsilon\mu}{2}} \left[ \sqrt{1 + \left(\frac{\sigma}{\epsilon\omega}\right)^2} - 1 \right]^{1/2}$$

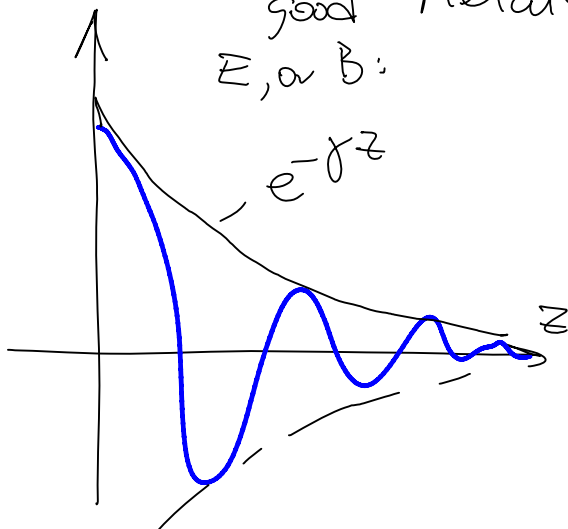
General Solutions become:

$$\vec{E}(z,t) = \vec{E}_0 \cdot e^{-\gamma z} \cdot e^{i(kz - \omega t)}$$

$$\vec{B}(z,t) = \vec{B}_0 \cdot e^{-\gamma z} \cdot e^{i(kz - \omega t)}$$

Both  $|\vec{E}|$  and  $|\vec{B}|$  are decaying expon. as a function of  $z$ , with decay "skin depth",

$$d \equiv \frac{1}{\gamma} \quad (\sim 10 \text{ nm for good metals})$$



\* Check back to Maxwell Eq's.

$$(i) \Rightarrow \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = 0$$

Since we assume no x-y dependence (as in page 5 of Lecture #1 notes)

$$\frac{\partial E_x}{\partial x} = \frac{\partial E_y}{\partial y} = 0 \Rightarrow \frac{\partial E_z}{\partial z} = i\vec{k} E_z = 0$$

or  $E_z = 0$ . Same for B(ii)

Waves are still transverse!

Let's define  $\hat{x}$  direction so that  $\vec{E} \parallel \hat{x}$

$$\vec{E}(z,t) = \vec{E}_0 e^{-\gamma z} \cdot e^{i(kz - \omega t)} \cdot \hat{x}$$

From (iii):

$$\begin{pmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \vec{E}_x & 0 & 0 \end{pmatrix} = \hat{y} \frac{\partial \vec{E}_x}{\partial z} - \hat{z} \frac{\partial \vec{E}_x}{\partial y} = \hat{y} \frac{\partial \vec{E}_x}{\partial z} = -\frac{\partial \vec{B}}{\partial t}$$

0, since  $\vec{E}$  has no y-dep.

If we assume B solution depends on time  $B \sim e^{-i\omega t}$ , then

$$\frac{\partial \vec{B}}{\partial t} = -i\omega \vec{B}$$

$$\text{Also, } \frac{\partial \vec{E}_x}{\partial z} = (ik - \gamma) \vec{E} = i\vec{k} \vec{E}$$

Therefore from (iii) above:

$$\vec{B} = \frac{\vec{k}}{\omega} \vec{E}_0 \cdot e^{-\gamma z} \cdot e^{i(kz - \omega t)} \cdot \hat{y}$$

So  $\vec{E} \perp \vec{B}$ , and  $\vec{E}, \vec{B} \perp \vec{k}$  (along  $\hat{z}$ )  
just like before.

But!  $|\vec{E}|$  and  $|\vec{B}|$  are out of phase, since  $\tilde{E}$  is complex!

If  $\tilde{K} = \sqrt{k^2 + \gamma^2} \cdot e^{i\phi}$ , then:

$$\vec{E}(z, t) = E_0 e^{-\gamma z} \cdot \cos(kz - \omega t + \delta_E) \hat{x}$$

$$\vec{B}(z, t) = \frac{E_0 \sqrt{k^2 + \gamma^2}}{\omega} e^{-\gamma z} \cdot \cos(kz - \omega t + \delta_E + \phi) \hat{y}$$

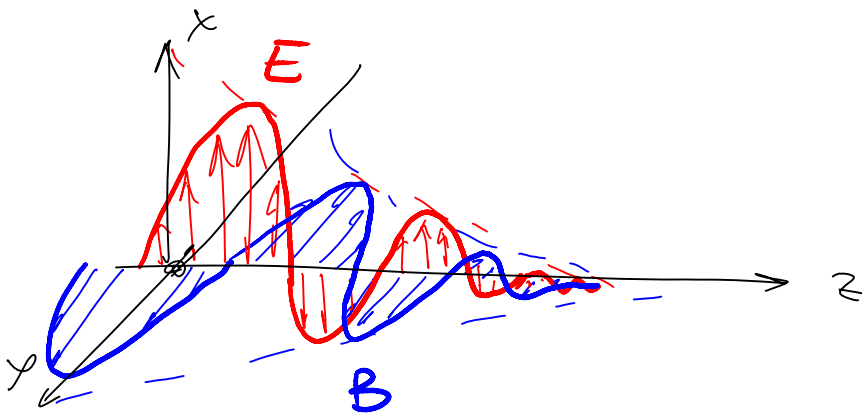
$E_0$  is now REAL ( $\tilde{E}_0 = E_0 \cdot e^{i\delta_E}$ )

Before we had the same phase for  $\vec{E}$  and  $\vec{B}$  ( $\delta_E = \delta_B$ )

which we "swept under the rug" by offsetting  $t=0$  by  $\Delta t = \frac{\delta_E}{\omega}$

Now we have additional phase  $\phi$ , where:

$$\phi = \arctan\left(\frac{\gamma}{k}\right) \quad (\text{just like Prob 9.3})$$



Phase shift is due to surface currents

## \* Reflection/transmission at conducting interface

So what happened to EM wave if it can't get through?  
It gets reflected!

Boundary conditions:

$$\begin{aligned} \epsilon_1 E_1^\perp - \epsilon_2 E_2^\perp &= \sigma \quad (i) & \epsilon_1, \mu_1 & \left| \begin{array}{l} \epsilon_2, \mu_2 \\ \text{---} \end{array} \right. \\ B_1^\perp - B_2^\perp &= 0 \quad (ii) & \text{---} & \xrightarrow{k} \text{---} \rightarrow T \\ E_1^\parallel - E_2^\parallel &= 0 \quad (iii) & R \xleftarrow{k} & \left| \begin{array}{l} \text{---} \\ \text{---} \end{array} \right. \\ \frac{B_1^\parallel}{\mu_1} - \frac{B_2^\parallel}{\mu_2} &= \vec{K} \times \hat{n} \quad (iv) & \text{---} & \left| \begin{array}{l} \text{---} \\ \text{---} \end{array} \right. \end{aligned}$$

(1) | (2)

For conductors  $\vec{K}$  (surface current per unit length) must be equal to 0, otherwise

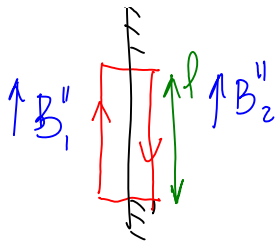
$\mathcal{J}$  (current per area)  $\rightarrow \infty$

and  $E = \frac{\mathcal{J}}{\sigma} \rightarrow \infty$  as well

\* (We can have finite current densities  $\mathcal{J}$ , but then total current per unit length can be made  $\rightarrow 0$

by tightening the  
Ampere's Loop





by tightening the  
Ampere's Loop  
 $dz \rightarrow 0, |\vec{K}| = \vec{j} \cdot dz \cdot \ell \rightarrow 0$



Just like in Lectures 2,3:

Incident wave:

$$E_I = \tilde{E}_{oI} e^{i(k_1 z - \omega t)}; B_I = \frac{\tilde{E}_{oI}}{v_1} e^{i(k_1 z - \omega t)}$$

Reflected:

$$E_R = \tilde{E}_{oR} e^{i(-k_1 z - \omega t)}; B_R = -\frac{\tilde{E}_{oR}}{v_1} e^{i(-k_1 z - \omega t)}$$

Transmitted wave has the  
solution derived above:

$$E_T = \tilde{E}_{oT} e^{i(\tilde{k}_2 z - \omega t)}; B_T = \frac{\tilde{k}_2 \tilde{E}_{oT}}{\omega} e^{i(\tilde{k}_2 z - \omega t)}$$

Same as before, but  $\tilde{k}$  is complex.

Boundary conditions:

(i) & (ii) yield nothing

since  $E_{\perp} = B_{\perp} = 0$  (ALSO means  $\vec{\sigma} = 0$ )

$$(iii): \tilde{E}_{oI} + \tilde{E}_{oR} = \tilde{E}_{oT}$$

$$(iv): \frac{\tilde{E}_{oI}}{\mu_1 v_1} - \frac{\tilde{E}_{oR}}{\mu_1 v_1} = \frac{\tilde{k}_2}{\mu_2 \omega} \tilde{E}_{oT}$$

Solutions are:

$$\tilde{E}_{oR} = \left( \frac{1 - \tilde{\beta}}{1 + \tilde{\beta}} \right) \tilde{E}_{oI}$$

$$\tilde{E}_{oT} = \left( \frac{2}{1 + \tilde{\beta}} \right) \tilde{E}_{oI}$$



$$\tilde{E}_{0T} = \left( \frac{2}{1 + \tilde{\beta}} \right) \tilde{E}_{0I}$$

(Same as before but  $\tilde{\beta} = \frac{\mu_1 \mu_2}{\mu_2 \omega} \tilde{K}_2$   
is complex now!)

Perfect conductor  $\sigma \rightarrow \infty$   
 $\Rightarrow \tilde{K}_2 \rightarrow \infty$   
 $\Rightarrow \tilde{\beta} \rightarrow \infty$

and  
 $\tilde{E}_{0R} = -\tilde{E}_{0I} ; \tilde{E}_{0T} = 0$

$$R = \left( \frac{|\tilde{E}_{0R}|}{|\tilde{E}_{0I}|} \right)^2 = 1 \quad T = 0$$

100% reflection at normal incidence,  
reflected wave is 180° out of  
phase w.r. to incident.

Please verify this every day  
by looking at the mirror  
reflection (also check for vampires!)

## \* PHYSICS SUMMARY:

EM waves in conductors (metals)  
induce currents (confined to "skin depth"  
near the surface) which "screen" the

near the surface) which "screen" the wave from passing through conductor.

As a result, waves get reflected (primarily), giving rise to shiny objects, mirrors etc.