PHYS 100C Lecture 14

Proof that retarded potential (shown for $V(r,t)$) satisfy Wave Equation (formerly known as Poisson Eq):

$$\nabla^2 V + \frac{1}{c^2}\cdot\frac{\partial^2 V}{\partial t^2} = -\frac{\rho}{\epsilon_0}$$

$$\nabla V = \frac{1}{4\pi\epsilon_0}\int\left[(\nabla\rho)\frac{1}{\imath} + \rho\nabla\left(\frac{1}{\imath}\right)\right]d\tau'$$

$$\nabla(\imath) = \hat{\imath}$$ Why?

if we make $\hat{x} \parallel \hat{\imath}$

$$\nabla(\imath) = \hat{x}\cdot\frac{\partial}{\partial x}(x) = \hat{x} = \hat{\imath}$$

(Even more obvious in spherical coord.)

More cumbersome proof:

$$\frac{\partial}{\partial x}\left(\sqrt{x^2+y^2+z^2}\right) = \frac{1}{2}\cdot\frac{1}{\sqrt{x^2+y^2+z^2}}\cdot 2x$$

Same for $\frac{\partial}{\partial y}$, $\frac{\partial}{\partial z}$:

$$\nabla\imath = \frac{x\hat{x}+y\cdot\hat{y}+z\cdot\hat{z}}{\sqrt{x^2+y^2+z^2}} = \frac{\vec{r}}{|r|} = \hat{r}$$

$$\nabla\left(\frac{1}{\imath}\right) = -\frac{1}{\imath^2}\cdot\nabla\imath = -\frac{\hat{\imath}}{\imath^2}$$

(Since $\frac{\partial}{\partial x}\left(\frac{1}{\imath}\right) = -\frac{1}{\imath^2}\cdot\frac{\partial \imath}{\partial x}$ etc.)

$$\nabla\rho = \hat{x}\cdot\frac{\partial\rho}{\partial x} + \ldots$$

$$\frac{\partial\rho}{\partial x} = \frac{\partial\rho}{\partial t_r}\cdot\frac{\partial t_r}{\partial x} = -\frac{\partial\rho}{\partial t_r}\cdot\frac{1}{c}\cdot\frac{\partial r}{\partial x} \quad \left(t_r = t - \frac{r}{c}\right)$$

$$\frac{\partial t}{\partial x} = \frac{\partial t}{\partial t_r}\cdot \partial x \quad \partial t_r \quad c \quad \partial x$$

$$\nabla\rho = \underbrace{\frac{\partial\rho}{\partial t_r}}_{\dot\rho}\cdot\left(-\frac{1}{c}\right)\underbrace{\nabla r}_{\hat{\mathscr{r}}} = -\frac{\dot\rho}{c}\cdot\hat{\mathscr{r}}$$

$$\nabla V = \frac{1}{4\pi\epsilon_0}\int\left(-\frac{\dot\rho}{c}\cdot\frac{\hat{\mathscr{r}}}{\mathscr{r}} - \rho\,\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right)d\tau'$$

$$\nabla^2 V = \frac{1}{4\pi\epsilon_0}\int\left[-\frac{1}{c}\left(\frac{\hat{\mathscr{r}}}{\mathscr{r}}\cdot\nabla\dot\rho + \underbrace{\dot\rho\cdot\nabla\cdot\left(\frac{\hat{\mathscr{r}}}{\mathscr{r}}\right)}\right) - \left(\underbrace{\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\cdot(\nabla\rho)} + \rho\nabla\cdot\left(\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right)\right)\right]d\tau'$$

cancel out

$$-\frac{1}{c}\dot\rho\,\underbrace{\nabla\cdot\left(\frac{\hat{\mathscr{r}}}{\mathscr{r}}\right)}_{1/\mathscr{r}^2} - \frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\underbrace{(\nabla\rho)}_{-\frac{\dot\rho}{c}\hat{\mathscr{r}}} = -\frac{1}{c}\dot\rho\,\frac{1}{\mathscr{r}^2} + \frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\cdot\frac{\dot\rho}{c}\cdot\hat{\mathscr{r}} = 0$$

$$\nabla\dot\rho = -\frac{\ddot\rho}{c}\,\hat{\mathscr{r}}$$

and $\nabla\cdot\left(\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right) = 4\pi\delta^3(\vec{\mathscr{r}})$

$$\nabla^2 V = \frac{1}{4\pi\epsilon_0}\int\left(\frac{\ddot\rho}{c^2\mathscr{r}} - 4\pi\rho\,\delta^3(\vec{\mathscr{r}})\right)d\tau'$$

$$\int\rho\cdot\delta^3(\vec{\mathscr{r}})\,d\tau' = \rho(\vec{\mathscr{r}}=0)$$

$$\frac{1}{4\pi\epsilon_0}\int\frac{\ddot\rho}{c^2\mathscr{r}}\cdot d\tau' = \frac{1}{c^2}\cdot\frac{\partial^2}{\partial t^2}\underbrace{\left(\frac{1}{4\pi\epsilon_0}\int\frac{\rho}{\mathscr{r}}\,d\tau'\right)}_{V}$$

$$\nabla^2 V = \frac{1}{c^2}\cdot\frac{\partial^2 V}{\partial t^2} - \frac{\rho}{\epsilon_0}$$

$$\Box^2 V = -\frac{\rho}{\epsilon_0}$$

* Question

V at observer position, time t
as a function of ρ at time $t_R = t - r/c$
where r is distance from source
to observer. When we derived $\nabla^2 V$
as a function of ρ, are they still
evaluated at time t and t_R, resp.?

Answer: when we did $\int \rho(\vec{r}, t_R) \cdot \delta^3(\vec{r})\, d\tau' =$
$= \rho(\vec{r}=0, t_R) \cdot 4\pi$

$\vec{r}=0$ means that ρ is evaluated at observer
position ($\vec{r}_{OA} = \vec{r}_{OB}$ and $\vec{r}_{AB} = \vec{0}$).

Therefore $t = t_R$ $\left(r/c = 0\right)$

$$\left(\nabla^2 - \frac{1}{c^2} \cdot \frac{\partial^2}{\partial t^2}\right) V(\vec{r}_{OB}, t) = -\frac{\rho(\vec{r}_{OA}, r_{AB}=0, t_R)}{\varepsilon_0}$$

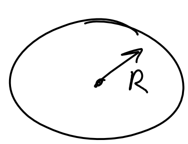
or:

$$\left(\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2}\right) V(\vec{r}, t) = -\frac{\rho(\vec{r}, t)}{\varepsilon_0}$$

* Question: why is $\nabla \cdot \left(\frac{\hat{r}}{r^2}\right) = 4\pi\, \delta^3(r)$

A: Divergence (Greene's) theorem:

$$\underset{\text{Volume}}{\int} (\nabla \cdot g) \cdot d\tau = \underset{\text{Area}}{\oint} g \cdot dS$$



(See 1.3.4 & 1.5.1)

$$g = \frac{\hat{r}}{r^2} \Rightarrow \oint_R g \cdot dS = \frac{1}{R^2} \cdot 4\pi R^2 = 4\pi$$

Independent of R!
Source of "field" exists
only at $R=0$.

Just like charge "4π"
centered at $R=0$,
infinitely small.

$$\int_{\text{volume}} (\nabla \cdot g) \cdot d\tau' = 4\pi$$

for any infitesmall volume
that includes $R=0$

$$\nabla \cdot \left(\frac{\hat{\tau}}{\tau^2}\right) = \delta^3(\vec{\tau}) \cdot 4\pi$$

Similarly, FOR $\nabla \cdot \left(\frac{\hat{\tau}}{\tau}\right)$:

$$g = \frac{\hat{\tau}}{\tau}$$

Area integral over sphere R:

$$\oint_R g \cdot dS = \frac{4\pi R^2}{R} = 4\pi R$$

$$\int_{\text{volume}} (\nabla \cdot g) \cdot d\tau' = 4\pi R$$

$$d\tau = 4\pi \tau^2 \cdot d\tau$$

$$\int_0^R (\nabla \cdot g) \cdot 4\pi \tau^2 \cdot d\tau = 4\pi R$$

$$(\nabla \cdot g) = \frac{1}{\tau^2}, \text{ then}$$

$$\int_0^R \frac{1}{\tau^2} \cdot 4\pi \tau^2 \cdot d\tau = \int_0^R 4\pi \cdot dr = 4\pi R$$

* Deep, far-out observation:

$$\Box^2 = \nabla^2 - \frac{\partial^2}{c^2 \partial t^2} \quad \text{is symmetrical}$$

w.r. to $t' = -t$ substitution.

Reversing the "arrow of time"

is not changing anything!
As a result, advanced potentials $t_A = t + r/c$ give a solution, just like retarded potentials.
PRACTICALLY, CAUSALITY AND ENTROPY DEFINES "ARROW OF TIME".