

PHYS-100C, Lectures 9-11 (WAVEGUIDES)

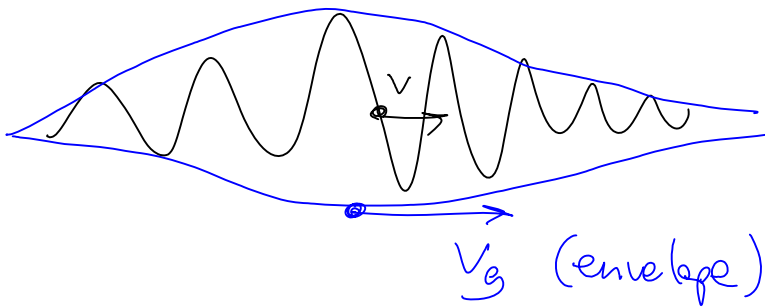
Monday, April 19, 2010
6:30 AM

* Dispersion:

$\epsilon (\Rightarrow n, v)$ depends on ω

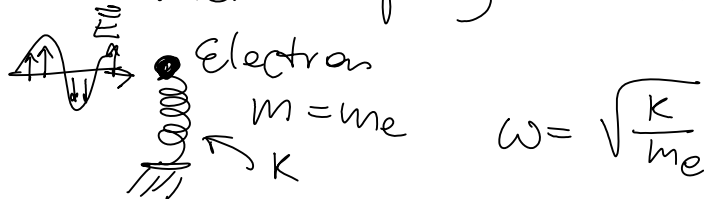
Phase velocity $v = \frac{\omega}{k}$

Group velocity $v_g = \frac{\partial \omega}{\partial k}$



Why does ϵ (and v, n etc.) depend on ω ?

Consider "spring" model of electron:



Eq. of motion:

$$m \frac{\partial^2 x}{\partial t^2} + m \gamma \frac{\partial x}{\partial t} + m \omega_0^2 x = e E_0 \cos \omega t$$

$\underbrace{\hspace{1.5cm}}_{F=ma} \quad \underbrace{\hspace{1.5cm}}_{\text{dissipation}} \quad \underbrace{\hspace{1.5cm}}_{kx} \quad \underbrace{\hspace{1.5cm}}_{\text{driving force}}$

General Solution: $x = \tilde{x}_0 \cdot e^{-i\omega t}$

$$-\omega^2 x - i\gamma x + \omega_0^2 x = \frac{eE_0}{m} e^{-i\omega t}$$

$$\tilde{x}_0 = \frac{e/m}{\omega_0^2 - \omega^2 - i\gamma} E_0$$

Polarization $\tilde{p} = e\tilde{x}$

$$\tilde{P} = \sum_{\text{ALL ELECTRONS/VOLUME}} \tilde{p} = \epsilon_0 \tilde{\chi}_e \tilde{E}$$

$$\tilde{E} = \epsilon_0 (1 + \tilde{\chi}_e)$$

$$\tilde{\epsilon}_R = 1 + \frac{Ne^2}{m\epsilon_0} \sum_j \frac{f_j}{\omega_j^2 - \omega^2 - i\gamma\omega}$$

Planar wave $E = E_0 \cdot e^{i\tilde{k}z - \omega t}$

where $\tilde{k} = \sqrt{\epsilon} k$, $\omega = k + i \frac{\alpha}{2}$

$$E = E_0 \cdot e^{-\frac{\alpha z}{2}} \cdot e^{i(kz - \omega t)}$$

α is absorption coefficient

$$n = \frac{ck}{\omega}$$

for small dispersive corrections:

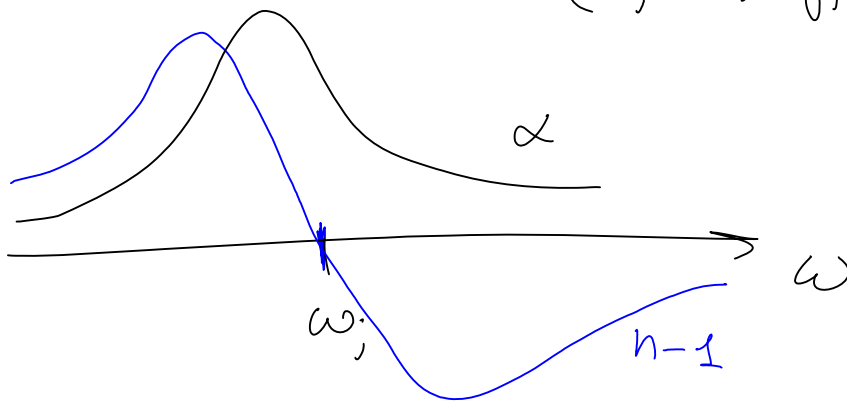
$$\sqrt{1+\epsilon} \approx 1 + \epsilon/2$$

$$n \approx \omega^{-1} \cdot \omega_0^2 = f \quad \gamma$$

$$\tilde{\kappa} = \frac{\omega}{c} \left[1 + \frac{Ne^2}{2m\epsilon_0} \sum_j \frac{f_j}{\omega_j^2 - \omega^2 - i\gamma_j\omega} \right]$$

$$n = \frac{ck}{\omega} \approx 1 + \frac{Ne^2}{2m\epsilon_0} \sum_j \frac{f_j (\omega_j^2 - \omega^2)}{(\omega_j^2 - \omega^2)^2 + \gamma_j^2 \omega^2}$$

$$\text{and } \alpha \approx \frac{Ne^2 \omega^2}{m\epsilon_0 c} \sum_j \frac{f_j \gamma_j}{(\omega_j^2 - \omega^2)^2 + \gamma_j^2 \omega^2}$$



Far away from resonances,

$$n \sim 1 + \sum \frac{\dots}{\omega_j^2 - \omega^2}$$

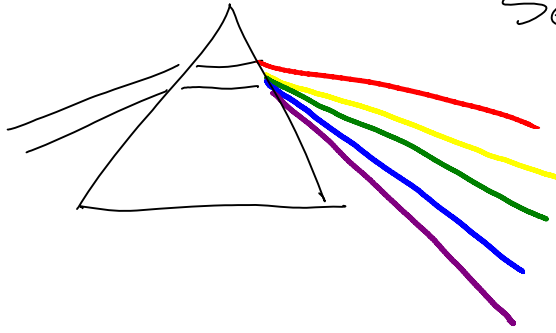
ω_j are in UV range

for visible light $\omega < \omega_j$,

$$\frac{1}{\omega_j^2 - \omega^2} = \frac{1}{\omega_j^2 (1 - \omega^2/\omega_j^2)} \approx \frac{1}{\omega_j^2} \left(1 + \frac{\omega^2}{\omega_j^2} \right)$$

$$n = 1 + A \left(1 + \frac{B}{\lambda^2} \right) \quad \text{Cauchy's Formula}$$

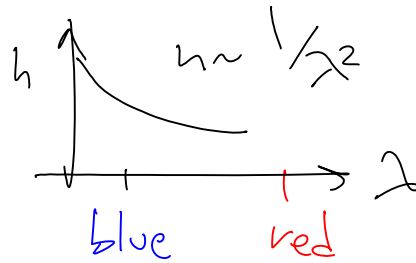
* Physics consequences:



See Cover ART
"Dark side
of the MOON"
by PINK FLOYD

n increases for shorter wavelengths
(blue/violet)

RAINBOWS:



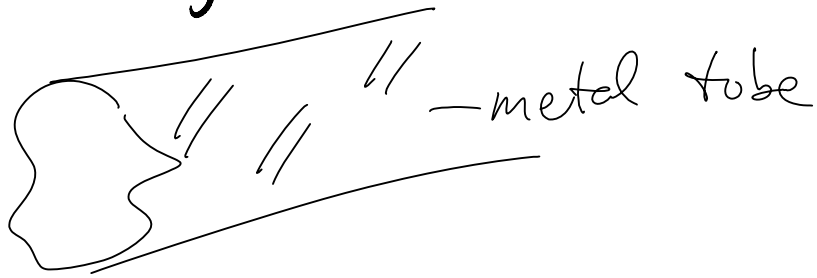
UV light

(bad for skin, can damage DNA)
gets substantially absorbed by
atmosphere (since $\omega_j \approx \text{UV}$)

For very large frequencies

$$\omega > \omega_j; \quad n < 1 \quad (\text{X-RAYS})$$

* Waveguides



Perfect Conductor:

$$E=0, B=0 \text{ inside}$$

Boundary conditions:

$$\epsilon_1 E_1^\perp - \epsilon_2 E_2^\perp = \sigma \quad (i)$$

$$B_1^\perp = B_2^\perp \quad (ii)$$

$$E_1^\parallel = E_2^\parallel \quad (iii)$$

$$\frac{1}{\mu_1} B_1^\parallel - \frac{1}{\mu_2} B_2^\parallel = K_f \times n \quad (iv)$$

$$\text{OR } E^\parallel = 0, B^\perp = 0 \quad (ii), (iii)$$

(since $E=0, B=0$ inside)

In interior of the waveguide

$$\nabla \cdot \vec{E} = 0 \quad \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \cdot \vec{B} = 0 \quad \nabla \times \vec{B} = \frac{1}{c^2} \frac{\partial \vec{E}}{\partial t}$$

No free charges, currents,
assume vacuum (thus ϵ_0, μ_0)

$$\vec{E} = \vec{E}_0 \cdot e^{i(kz - \omega t)}$$

$$\vec{B} = \vec{B}_0 \cdot e^{i(kz - \omega t)}$$

$$\vec{B} = \vec{B}_0 \cdot e^{i(kz - \omega t)}$$

E, B are not generally transverse (which was a result of plane waves with no $x-y$ dependence for E, B)

$$\nabla \times \vec{E} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix} = \hat{x} \left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} \right) + \hat{y} \left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} \right) + \hat{z} \left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} \right)$$

For (iii):

$$(z): \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = i\omega B_z \quad (1)$$

$$(x): \frac{\partial E_z}{\partial y} - ik E_y = i\omega B_x \quad (2)$$

$$(y): ik E_x - \frac{\partial E_z}{\partial x} = i\omega B_y \quad (3)$$

$$\left(\text{Since } \frac{\partial E_y}{\partial z} = ik E_y \quad \frac{\partial E_x}{\partial z} = ik E_x \right)$$

Similarly for Eq.(IV):

$$(z): \frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} = -\frac{i\omega}{c^2} E_z \quad (4)$$

$$(x): \frac{\partial B_z}{\partial y} - ik B_y = -\frac{i\omega}{c^2} E_x \quad (5)$$

$$(y): ik B_x - \frac{\partial B_z}{\partial x} = -\frac{i\omega}{c^2} E_y \quad (6)$$

From (3) & (5), eliminate B_y :

$$E_x = \frac{i}{(\omega/c)^2 - k^2} \left(k \frac{\partial E_z}{\partial x} + \omega \frac{\partial B_z}{\partial y} \right)$$

From (2) & (6), eliminate B_x

$$E_y = \frac{i}{(\omega/c)^2 - k^2} \left(k \frac{\partial E_z}{\partial y} - \omega \frac{\partial B_z}{\partial x} \right)$$

Similarly for B_x, B_y :

(2) and (6):

$$B_x = \frac{i}{(\omega/c)^2 - k^2} \left(k \frac{\partial B_z}{\partial x} - \frac{\omega}{c^2} \frac{\partial E_z}{\partial y} \right)$$

$$B_y = \frac{i}{(\omega/c)^2 - k^2} \left(k \frac{\partial B_z}{\partial y} + \frac{\omega}{c^2} \frac{\partial E_z}{\partial x} \right)$$

E_z & B_z determine everything else.

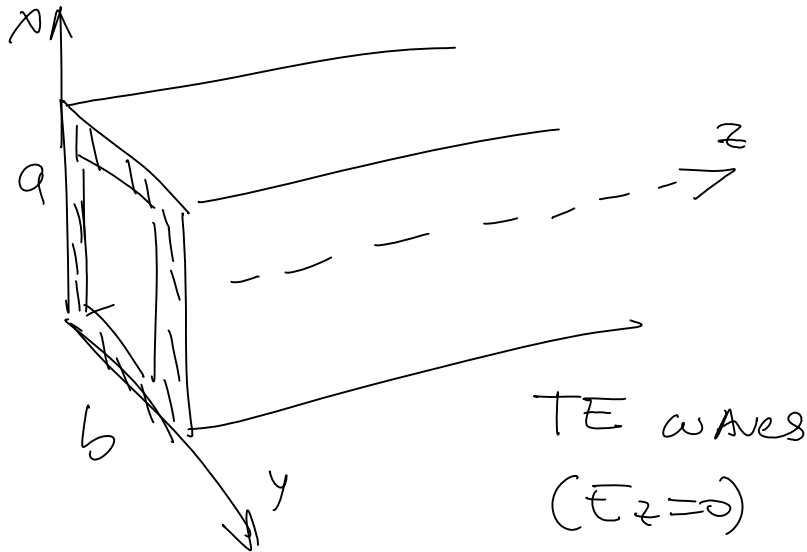
$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \left(\frac{\omega}{c} \right)^2 - k^2 \right) \begin{matrix} E_z \\ B_z \end{matrix} = 0$$

$$E_z = 0 \Rightarrow \text{TE waves}$$

$$B_z = 0 \Rightarrow \text{TM waves}$$

$$E_z = B_z = 0 \Rightarrow \text{TEM waves} \\ (\text{cannot occur in hollow WG})$$

* Rectangular Waveguide:



$$E_z, B_z(x, y) = X(x) Y(y)$$

$$Y \frac{\partial^2 X}{\partial x^2} + X \frac{\partial^2 Y}{\partial y^2} + \left[\left(\frac{\omega}{c} \right)^2 - k^2 \right] XY = 0$$

$$\frac{1}{X} \frac{\partial^2 X}{\partial x^2} = -k_x^2 \quad \frac{1}{Y} \frac{\partial^2 Y}{\partial y^2} = -k_y^2$$

$$k_x^2 + k_y^2 = \left(\frac{\omega}{c} \right)^2 - k^2$$

$$X = A \sin(k_x \cdot x) + B \cdot \cos(k_x \cdot x)$$

$$\text{at } x=0 ; x=a \quad B_x = 0$$

$$B_x = \dots \frac{\partial B_z}{\partial x} \dots \frac{\partial E_z}{\partial y} = 0$$

$$\parallel$$

$$\frac{\partial X}{\partial x} \cdot Y = 0 \quad (TE)$$

$$\frac{\partial X}{\partial x} = k_x (A \cdot \cos(k_x x) - B \sin(k_x x)) = 0$$

for $x=0, a \Rightarrow A=0$

$$\sin(k_x x) = 0 \quad \text{at } x=a$$

$$k_x = \frac{m\pi}{a} \quad m=0,1,2,\dots$$

Similarly $k_y = \frac{h\pi}{b} \quad h=0,1,2,\dots$

$$B_z = XY = B_0 \cdot \cos \frac{m\pi x}{a} \cdot \cos \frac{h\pi y}{b}$$

TE_{mn} mode

Standing waves (2D) in xy

$$k = \sqrt{\left(\frac{\omega}{c}\right)^2 - \left(\frac{\pi m}{a}\right)^2 - \left(\frac{\pi h}{b}\right)^2}$$

$$\omega_{mn}^2 = \left(\frac{\pi m c}{a}\right)^2 + \left(\frac{\pi h c}{b}\right)^2$$

cut-off frequency for TE_{mn}

If $\omega < \omega_{mn}$, k is complex
(attenuation)

$$k = \frac{1}{c} \sqrt{\omega^2 - \omega_{mn}^2}$$

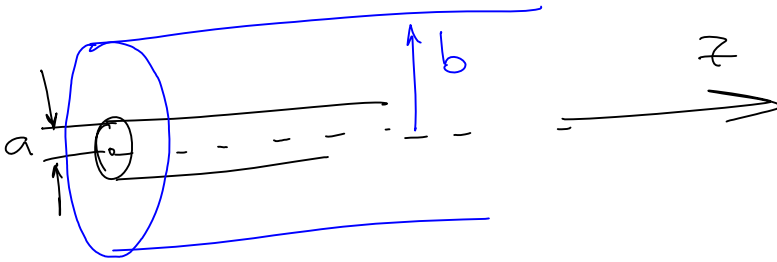
$$v = \frac{\omega}{k} = \frac{c}{\sqrt{1 - \left(\frac{\omega_{mn}}{\omega}\right)^2}} \quad (> c)$$

↪ . . .

Group velocity:

$$V_g = \frac{\partial \omega}{\partial k} = \frac{1}{\partial k_{\omega}} = c \sqrt{1 - \left(\frac{\omega_{mn}}{\omega}\right)^2} < c$$

* Coaxial cable:



Solution allows TEM
($B_z = E_z = 0$)

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = i\omega B_z = 0$$

$$\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} = -\frac{i\omega}{c^2} E_z = 0$$

$$\nabla \cdot \vec{E} = 0 \Rightarrow \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} = 0$$

$$\nabla \cdot \vec{B} = 0 \Rightarrow \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} = 0$$

Electro/Magneto-statics:

$$\vec{E} = \frac{A \cdot \cos(kz - \omega t)}{r} \cdot \hat{r}$$

$$\vec{B} = \frac{A \cdot \cos(kz - \omega t)}{cr} \cdot \hat{\phi}$$