PHYS 100C, Lecture 13

Monday, April 26, 2010
6:30 AM

* Potential & Fields

In electrostatics:

$$\nabla \cdot E = \frac{\rho}{\varepsilon_0} \qquad \nabla \times E = -\frac{\partial B}{\partial t} = 0 \quad (B = \text{const})$$

This allowed introducing scalar potential V,

$$\vec{E} = -\vec{\nabla} V$$

Since $\bar{\nabla} \times \bar{E} = -\bar{\nabla} \times (\bar{\nabla} V) = 0$

No longer works if $\partial B / \partial t \neq 0$

$\bar{\nabla} \cdot \bar{B} = 0$ so vector potential still works:

$$\bar{B} = \bar{\nabla} \times \bar{A}$$

then $\nabla \times E = -\frac{\partial B}{\partial t}$ becomes:

$$\nabla \times E = -\frac{\partial}{\partial t}(\nabla \times A) \quad \text{or}$$

$$\nabla \times \left(E + \frac{\partial A}{\partial t}\right) = 0$$

New quantity, $E' = E + \frac{\partial A}{\partial t} = -\nabla V$

since $\nabla \times E' = 0$

then $E = -\nabla V - \frac{\partial A}{\partial t}$

Gauss' Law: $\nabla^2 V + \frac{\partial}{\partial t}(\nabla \cdot A) = -\frac{\rho}{\varepsilon_0}$

similar to Poisson Eq. Except for this

last Maxwell Eq:

$$\left(\nabla^2 A - \mu_0 \varepsilon_0 \cdot \frac{\partial^2}{\partial t^2}\right) - \nabla\left(\bar{\nabla} \cdot \bar{A} + \mu_0 \varepsilon_0 \frac{\partial V}{\partial t}\right) = -\mu_0 J$$

Monstrous, UGLY equation. . . .

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But, reduced 6 unknowns E_x, E_y, E_z B_x, B_y, B_z

to only 4 (V, A_x, A_y, A_z).

Useful for other purposes.

A & V not unique:

can introduce new

$$A' = A + \alpha$$
$$V' = V + \beta$$

that give rise to the same $\vec{E}(\vec{r}, t)$ $\vec{B}(\vec{r}, t)$

Called GAUGE TRANSFORMATIONS

* Gauge Transform.

$$\left.\begin{aligned} A' &= A + \alpha \\ V' &= V + \beta \end{aligned}\right\} \text{ give the same } E, B \text{ as } A, V$$

$$B = \nabla \times A = \nabla \times A' \Rightarrow \nabla \times \alpha = 0$$

$\alpha = \nabla \lambda$ where λ is scalar

$$E = -\nabla V - \frac{\partial A}{\partial t} = -\nabla V' - \frac{\partial A'}{\partial t} \Rightarrow$$

$$\nabla \beta + \frac{\partial \alpha}{\partial t} = 0 \quad \text{OR} \quad \alpha = \nabla \lambda$$

$$\nabla \left(\beta + \frac{\partial \lambda}{\partial t} \right) = 0$$

Solution: $\beta = -\frac{\partial \lambda}{\partial t}$

$$\left.\begin{aligned} A' &= A + \nabla \lambda \\ V' &= V - \frac{\partial \lambda}{\partial t} \end{aligned}\right\} \text{ Give identical } E, B \text{ for any scalar field } \lambda$$

Gauge transformations

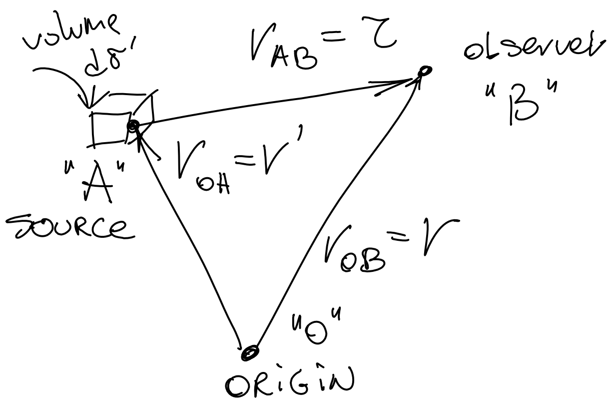
* Coulomb Gauge:

$$\nabla \cdot A = 0 \Rightarrow \nabla^2 V = -\frac{\rho}{\varepsilon_0}$$

Solution is

$$V(r,t) = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho(r',t)}{\gamma} \, d\tau'$$

* Note: three r's:



$$V(r_{OB}, t) = \ldots \int \frac{\rho(r_{OA}, t)}{r_{OB}} \, d\tau$$

Physics Problem with Coulomb gauge:

V is instantaneous (depends on "right now" charge density $\rho(r_{OA}, t)$, far away).

E is "retarded" or delayed, by: $E = -\nabla V - \frac{\partial A}{\partial t}$, this part this part

* Far Superior, Awesomer Gauge:

Lorentz Gauge:

$$\nabla \cdot A = -\mu_0 \varepsilon_0 \frac{\partial V}{\partial t}$$

to kill of the ugly 2^{nd} part in:

$$\Big(\nabla^2 A - \underbrace{\mu_0 \varepsilon_0}_{1/c^2} \frac{\partial^2 A}{\partial t^2}\Big) - \nabla \underbrace{\Big(\nabla \cdot A + \mu_0 \varepsilon_0 \frac{\partial V}{\partial t}\Big)}_{=0!} = -\mu_0 J$$

Remaining part:

$$\Big(\nabla^2 - \frac{1}{c^2} \cdot \frac{\partial^2}{\partial t^2}\Big) A = -\mu_0 J$$

and $\nabla^2 V + \frac{\partial}{\partial t}(\nabla \cdot A) = -\frac{\rho}{\varepsilon_0}$ becomes

$$\Big(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t}\Big) V = -\frac{\rho}{\varepsilon_0}$$

Introduce d'Alembertian:

$$\Box^2 = \nabla^2 - \frac{1}{c^2} \cdot \frac{\partial^2}{\partial t^2} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial (ict)^2}$$

("ict" is complex four-dimensional space-time coordinate)

Maxwell's Eq's (1) and (4) reduced:

$$\Box^2 V = -\frac{\rho}{\varepsilon_0}$$

$$\Box^2 A = -\mu_0 J$$

(Eqs (2), (3) are already satisfied)

it takes time for light to travel distance $\mathscr{r}$:

potentials evaluated at "retarded" time $t_r \equiv t - \frac{\mathscr{r}}{c}$

$$V(r,t) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(r', t_r)}{\mathscr{r}} \, d\tau'$$

$$A(r,t) = \frac{\mu_0}{4\pi}\int \frac{J(r', t_r)}{\mathscr{r}} \, d\tau'$$

* does NOT work for $E(r,t)$ or $B(r,t)$ – by simply integrating Coulomb's & Biot-Savart's Laws evaluated at time $t_r = t - \mathscr{r}/c$

Proof that retarded potential (shown for $V(r,t)$) satisfy Wave Equation (formerly known as Poisson Eq):

$$\nabla^2 V + \frac{1}{c^2}\cdot\frac{\partial^2 V}{\partial t^2} = -\frac{\rho}{\varepsilon_0}$$

$$\nabla V = \frac{1}{4\pi\varepsilon_0}\int\left[(\nabla\rho)\frac{1}{\mathscr{r}} + \rho\nabla\left(\frac{1}{\mathscr{r}}\right)\right]d\tau'$$

$$\nabla(\mathscr{r}) = \hat{\mathscr{r}} \qquad \text{Why?}$$

if we make $\hat{x} \parallel \hat{\mathscr{r}}$

$$\nabla(\mathscr{r}) = \hat{x}\cdot\frac{\partial}{\partial x}(x) = \hat{x} = \hat{\mathscr{r}}$$

(Even more obvious in spherical coord.)

More cumbersome proof:

$$\frac{\partial}{\partial x}\left(\sqrt{x^2+y^2+z^2}\right) = \frac{1}{2}\cdot\frac{1}{\sqrt{x^2+y^2+z^2}}\cdot 2x$$

Same for $\frac{\partial}{\partial y}$, $\frac{\partial}{\partial z}$:

$$\nabla \mathscr{r} = \frac{x\hat{x}+y\cdot\hat{y}+z\cdot\hat{z}}{\sqrt{x^2+y^2+z^2}} = \frac{\vec{r}}{|r|} = \hat{r}$$

$$\nabla\left(\frac{1}{\mathscr{r}}\right) = -\frac{1}{\mathscr{r}^2}\cdot\nabla\mathscr{r} = -\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}$$

(Since $\frac{\partial}{\partial x}\left(\frac{1}{\mathscr{r}}\right) = -\frac{1}{\mathscr{r}^2}\cdot\frac{\partial \mathscr{r}}{\partial x}$ etc.)

$$\nabla \rho = \hat{x}\cdot\frac{\partial\rho}{\partial x} + \dots$$

$$\frac{\partial\rho}{\partial x} = \frac{\partial\rho}{\partial t_r}\cdot\frac{\partial t_r}{\partial x} = -\frac{\partial\rho}{\partial t_r}\cdot\frac{1}{c}\cdot\frac{\partial r}{\partial x} \qquad \left(t_r = t - \frac{r}{c}\right)$$

$$\nabla\rho = \underbrace{\frac{\partial\rho}{\partial t_r}}_{\dot\rho}\cdot\left(-\frac{1}{c}\right)\underbrace{\nabla r}_{\hat{\mathscr{r}}} = -\frac{\dot\rho}{c}\cdot\hat{\mathscr{r}}$$

$$\nabla V = \frac{1}{4\pi\varepsilon_0}\int\left(-\frac{\dot\rho}{c}\cdot\frac{\hat{\mathscr{r}}}{\mathscr{r}} - \rho\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right)d\tau'$$

$$\nabla^2 V = \frac{1}{4\pi\varepsilon_0}\int\left[-\frac{1}{c}\left(\frac{\hat{\mathscr{r}}}{\mathscr{r}}\cdot\nabla\dot\rho + \underbrace{\dot\rho\cdot\nabla\cdot\left(\frac{\hat{\mathscr{r}}}{\mathscr{r}}\right)}\right) - \left(\underbrace{\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\cdot(\nabla\rho)} + \rho\nabla\cdot\left(\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right)\right)\right]d\tau'$$

cancel out

$$-\frac{1}{c}\dot\rho\,\underbrace{\nabla\cdot\left(\frac{\hat{\mathscr{r}}}{\mathscr{r}}\right)}_{1/\mathscr{r}^2} - \frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\underbrace{(\nabla\rho)}_{-\frac{\dot\rho}{c}\hat{\mathscr{r}}} = -\frac{1}{c}\dot\rho\frac{1}{\mathscr{r}^2} + \frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\cdot\frac{\dot\rho}{c}\cdot\hat{\mathscr{r}} = 0$$

$$\nabla\dot\rho = -\frac{\ddot\rho}{c}\hat{\mathscr{r}}$$

and $\nabla\cdot\left(\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right) = 4\pi\delta^3(\vec{\mathscr{r}})$

$$\nabla^2 V = \frac{1}{4\pi\varepsilon_0}\int\left(\frac{\ddot\rho}{c^2\mathscr{r}} - 4\pi\rho\,\delta^3(\vec{\mathscr{r}})\right)d\tau'$$

$$\int\rho\cdot\delta^3(\vec{\mathscr{r}})\,d\tau' = \rho(\vec{\mathscr{r}}=0)$$

$$\frac{1}{4\pi\varepsilon_0}\int\frac{\ddot\rho}{c^2\mathscr{r}}\cdot d\tau' = \frac{1}{c^2}\cdot\frac{\partial^2}{\partial t^2}\underbrace{\left(\frac{1}{4\pi\varepsilon_0}\int\frac{\rho}{\mathscr{r}}\,d\tau'\right)}$$

∇

$$\nabla^2 V = \frac{1}{c^2} \cdot \frac{\partial^2 V}{\partial t^2} - \frac{\rho}{\varepsilon_0}$$

$$\nabla^2 V = -\frac{\rho}{\varepsilon_0}$$

* Question (from Jeff, Mike, Scott).

V at observer position, time t as a function of ρ at time $t_R = t - r/c$ where r is distance from source to observer. When we derived $\nabla^2 V$ as a function of ρ, are they still evaluated at time t and t_R, resp.?

ANSWER: when we did $\int \rho(\vec{r}, t_R) \cdot \delta^3(\vec{r}) \, d\tau' =$
$= \rho(\vec{r}=0, t_R) \cdot 4\pi$

$\vec{r} = 0$ means that ρ is evaluated at observer position ($\vec{r}_{OA} = \vec{r}_{OB}$ and $\vec{r}_{AB} = \vec{0}$).

Therefore $t = t_R$ ($r/c = 0$)

$$\left(\nabla^2 - \frac{1}{c^2} \cdot \frac{\partial^2}{\partial t^2}\right) V(\vec{r}_{OB}, t) = -\frac{\rho(\vec{r}_{OA}, r_{AB}=0, t_R)}{\varepsilon_0}$$

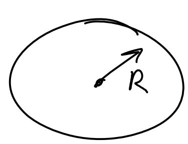
OR:

$$\left(\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2}\right) V(\vec{r}, t) = -\frac{\rho(\vec{r}, t)}{\varepsilon_0}$$

* Question: why is $\nabla \cdot \left(\frac{\hat{r}}{r^2}\right) = 4\pi \delta^3(r)$

A: DIVERGENCE (GREENE'S) THEOREM:

$$\int_{\text{VOLUME}} (\nabla \cdot g) \cdot d\tau = \oint_{\text{AREA}} g \cdot dS$$



(See 1.3.4 & 1.5.1)

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$$g = \frac{\hat{r}}{r^2} \Rightarrow \oint_R g \cdot dS = \frac{1}{R^2} \cdot 4\pi R^2 = 4\pi$$

Independent of R!

Source of "field" exists *only* at $R = 0$.

Just like charge "4π" centered at $R=0$, infinitely small.

$$\int_{\text{volume}} (\nabla \cdot g) \cdot d\tau' = 4\pi$$

for any infitesmall volume that includes $R=0$

$$\nabla \cdot \left(\frac{\hat{r}}{r^2}\right) = \delta^3(\vec{r}) \cdot 4\pi$$

Similarly, FOR $\nabla \cdot \left(\frac{\hat{r}}{r}\right)$:

$$g = \frac{\hat{r}}{r}$$

Area integral over sphere R:

$$\oint_R g \cdot dS = \frac{4\pi R^2}{R} = 4\pi R$$

$$\int_{\text{volume}} (\nabla \cdot g) \cdot d\tau' = 4\pi R$$

$$d\tau = 4\pi r^2 \cdot dr$$

$$\int_0^R (\nabla \cdot g) \cdot 4\pi r^2 \cdot dr = 4\pi R$$

$(\nabla \cdot g) = \frac{1}{r^2}$, then

$$\int_0^R \frac{1}{r^2} \cdot 4\pi r^2 \cdot dr = \int_0^R 4\pi \cdot dr = 4\pi R$$

$$\int_0^R \frac{1}{r^2}\cdot 4\pi r^2 \cdot dr = \int_0^R 4\pi \cdot dr = 4\pi R$$

* Deep, far-out observation:

$\Box^2 = \nabla^2 - \frac{\partial^2}{c^2 \partial t^2}$ is symmetrical

w.r. to $t' = -t$ substitution.

Reversing the "arrow of time" is not changing anything!

As a result, advanced potentials $t_A = t + r/c$ give a solution, just like retarded potentials.

PRACTICALLY, CAUSALITY AND ENTROPY DEFINES "ARROW OF TIME".