PHYS 100C, Lecture #10

Wednesday, April 29, 2009
9:24 PM

RADIATION:

Where do waves (from Chapter 9) come from? Accelerated charges.

Lienard-Wichert:

$$E \sim \ldots \underbrace{\frac{1}{(\imath u)^2}}_{\text{Scales as } \frac{1}{\imath^2}} + \ldots \underbrace{\frac{\imath\{\imath \times (u \times a)\}}{(\imath u)^3}}_{\text{Scales as } \frac{1}{\imath}}$$

(Coulomb-like) $\qquad \sim a$ (acceleration)

Radiation term

Because $B = \frac{\hat{\imath} \times \vec{E}}{c}$ (LW potentials),

$$B \sim \ldots \frac{1}{\imath^2} + \ldots \frac{a}{\imath}$$

Radiated intensity:

$$S \sim E \times B \sim \ldots \underbrace{\frac{a^2}{\imath^2}}_{\substack{\text{acceleration}\\ \text{term}}} + \ldots \underbrace{\frac{a}{\imath^3}}_{\substack{\text{cross-}\\ \text{terms}}} + \underbrace{\frac{\ldots}{\imath^4}}_{\substack{\text{Coulomb}\\ \text{term}}}$$

For large distances, integrated over a sphere of radius $\imath$, the acceleration term $\sim a^2/\imath^2$ survives, since total power radiated is $\sim a^2/\imath^2 \cdot 4\pi \imath^2 \sim a^2 = \text{const}$.

Higher-order contributions decay.

For harmonic oscillator

$$x = x_0 \cdot \cos \omega t$$

$$x' = -x_0 \cdot \omega \cdot \sin \omega t$$

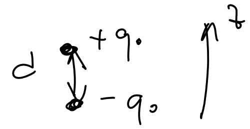
$$a = x'' = -\omega^2 x_0 \cdot \cos \omega t$$

Since $a \sim \omega^2$, radiated EM wave intensity $P \sim \langle S \rangle \sim \frac{a^2}{\imath^2} \sim \frac{\omega^4}{\imath^2}$

(HAND-WAVING ARGUMENT).

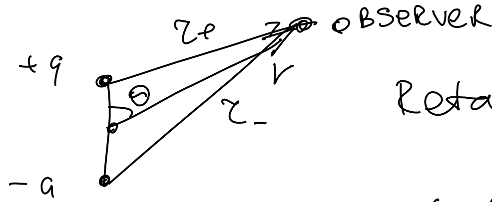
Here's more DETAILED DERIVATION.

Consider dipole oscillating at frequency ω:



$$\vec{p}(t) = p_0 \cdot \cos(\omega t) \cdot \hat{z}$$

$$p_0 = q_0 \cdot d$$



Retarded potentials:

$$V = \frac{q_0}{4\pi\varepsilon_0} \cdot \left[\frac{\cos\left[\omega\left(t - \frac{r_+}{c}\right)\right]}{r_+} - \frac{\cos\left[\omega\left(t - \frac{r_-}{c}\right)\right]}{r_-} \right]$$

$$r_\pm^2 = r^2 \mp r d \cos\theta + \left(\frac{d}{2}\right)^2$$

If $d \ll r \Rightarrow r_\pm = r\left(1 \mp \frac{d}{2r} \cdot \cos\theta\right)$

Also $\frac{1}{1+x} \approx 1 - x \Rightarrow \frac{1}{r_\pm} = \frac{1}{r}\left(1 \pm \frac{d}{2r}\cos\theta\right)$

$$\varphi = \frac{\Delta r \cdot \omega}{c} = \frac{d \cdot \cos\theta \cdot \omega}{2c} \simeq \frac{\pi d \cdot \cos\theta}{\lambda} \ll 1$$

since $\frac{\omega}{c} = \frac{2\pi}{\lambda}$ (or $d \ll \lambda$)

$$\cos(A+B) = \cos A \cdot \cos B - \sin A \cdot \sin B$$

$$\cos\left[\omega\left(t - \frac{r_\pm}{c}\right)\right] = \cos(\omega t - kr) \cdot \underbrace{\cos\varphi}_{\simeq 1 - \frac{\varphi^2}{2}} \mp$$

$$\mp \sin(\omega t - kr) \sin\varphi \approx$$

$$\approx \cos(\omega t - r/\lambda) \mp \varphi \cdot \sin(\omega t - kr)$$

(dropping terms $\sim \varphi^2$)

Plugging $\frac{1}{r_\pm}$ and $\cos(\omega t - kr_\pm)$ into expression for $V(r,t)$:

$$V = \frac{q_0}{4\pi\varepsilon_0}\left(\frac{\cos(\omega t-kr) - \sin(\omega t-kr)\frac{\pi \cdot d \cdot \cos\theta}{\lambda}}{r}\left(1+\frac{d\cos\theta}{2r}\right) - \frac{\cos(\omega t-kr)+\sin(\omega t-kr)\frac{\pi d \cdot \cos\theta}{\lambda}}{r}\left(1-\frac{d\cos\theta}{2r}\right)\right) =$$

$$= \frac{q_0}{4\pi\varepsilon_0 r}\Big(\cancel{\cos(\omega t-kr)} - \sin(\omega t-kr)\cdot\frac{\pi d\cdot\cos\theta}{\lambda} + \cos(\omega t-kr)\frac{d\cos\theta}{2r} +$$
$$-\sin(\omega t-kr)\cdot\frac{\pi d^2\cos^2\theta}{2r\lambda} - \cancel{\cos(\omega t-kr)} - \sin(\omega t-kr)\frac{\pi d\cos\theta}{\lambda} +$$
$$+\cos(\omega t-kr)\cdot\frac{d\cdot\cos\theta}{r} + \sin(\omega t-kr)\frac{\pi d^2\cos^2\theta}{2r\lambda}\Big)$$

We will neglect $\sim\frac{d^2}{r\lambda}$ term (2^{nd} order)

Zeroth order term $\cos(\omega t-kr)$ cancels out

Leading terms are 1^{st} ORDER: $\sim\frac{d}{\lambda}$ (blue) and $\sim\frac{d}{r}$ (green). Take d and $\cos\theta$ outside and use $p_0 = q_0 \cdot d$:

main term

$$V = \frac{p_0\cos\theta}{4\pi\varepsilon_0 r}\left\{-\frac{2\pi\cdot\sin(\omega t-kr)}{\lambda} + \frac{\cos(\omega t-kr)}{r}\right\}$$

Check: for $\omega\to 0$ $\quad V = \frac{p_0\cos\theta}{4\pi\varepsilon_0 r^2}$

(static dipole potential)

Two terms: $\sim\frac{1}{\lambda}$ and $\sim\frac{1}{r}$

For large $r \gg \lambda$, keep only $\frac{1}{\lambda}$ term

$$V = -\frac{p_0 \cdot \cos\theta}{4\pi\varepsilon_0 r \lambda} \sin(\omega t - kr)$$

Current in oscillating dipole:

$$I(t) = \frac{\partial q}{\partial t} \cdot \hat{z} = -q_0 \cdot \omega \cdot \sin\omega t \; \hat{z}$$

$$A = \frac{\mu_0}{4\pi} \int_{-d/2}^{d/2} \frac{-q_0 \omega \sin(\omega t - kr)}{r} \cdot \hat{z} \cdot dz$$

$$A = -\frac{\mu_0}{4\pi} \cdot \frac{p_0 \omega}{r} \sin(\omega t - kr) \hat{z}$$

$$\nabla V = \frac{\partial V}{\partial r} \cdot \hat{r} + \frac{1}{r} \frac{\partial V}{\partial \theta} \cdot \hat{\theta}$$

$$\frac{\partial V}{\partial r} = \ldots \underbrace{\cancel{\frac{\sin(\ldots)}{r^2}}}_{\text{small } (r \gg \lambda)} - \frac{\cos(\ldots)}{r\lambda}$$

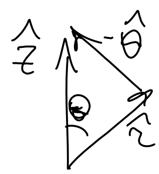
$$\frac{1}{r}\frac{\partial V}{\partial \theta} \sim \frac{1}{r^2}$$ (also small compared to $\sim \frac{1}{r}$ term

$$\nabla V = \frac{p_0}{4\pi\varepsilon_0 \lambda^2} \cdot \frac{\cos\theta}{r} \cdot \cos(\omega t - kr) \, \hat{r}$$

Similarly

$$\frac{\partial A}{\partial t} = -\frac{\mu_0}{4\pi} \cdot \frac{p_0 \omega^2}{r} \cos(\omega t - kr) \, \hat{z}$$

$$\hat{z} = \hat{r} \cdot \cos\theta - \sin\theta \cdot \hat{\theta}$$



$$E = -\nabla V - \frac{\partial A}{\partial t} = -\frac{\mu_0}{4\pi}\, p_0 \omega^2 \left(\frac{\sin\theta}{r}\right)\cdot\cos(\omega t - kr)\,\hat{\theta}$$

(the $\cos\theta$ terms in ∇V and $\frac{\partial A}{\partial t}$ cancel)

$$\nabla \times A = \frac{1}{r}\left[\frac{\partial}{\partial r}(r A_\theta) - \frac{\partial A_r}{\partial \theta}\right]\hat{\phi}$$

$$B = \nabla \times A = -\frac{\mu_0 p_0 \omega^2}{4\pi c}\cdot\left(\frac{\sin\theta}{r}\right)\cos(\omega t - kr)\,\hat{\phi}$$

(or, could have used

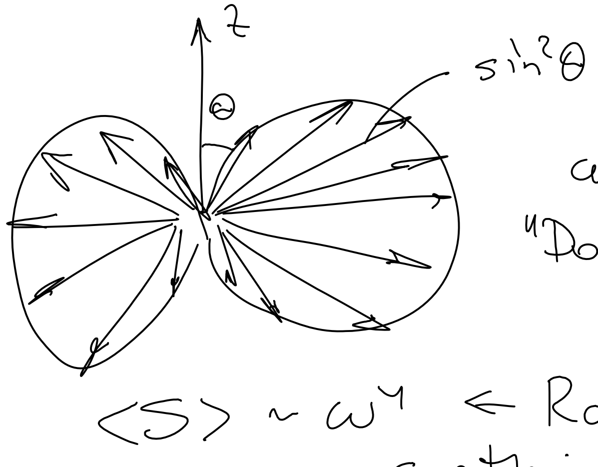
$$\vec{B} = \frac{\hat{r}}{c} \times \vec{E}$$, from LW potentials)

$\bar{E}, \bar{B}$ transverse ($\perp \hat{r}$, $\hat{E} \perp \bar{B}$), in phase, just like EM waves

$$\vec{S} = \frac{1}{\mu_0}\left[E \times B\right] = \frac{\mu_0}{c}\left(\frac{p_0\omega^2}{4\pi}\left(\frac{\sin\theta}{r}\right)\cos(\omega t - kr)\right)^2 \hat{r}$$

$$\langle S \rangle = \frac{\mu_0 p_0^2 \omega^4}{32\pi^2 c}\cdot\frac{\sin^2\theta}{r^2}\cdot\hat{r}$$

(since $\langle \cos^2(\omega t - r/\lambda)\rangle = 1/2$)



No scattering along dipole axis ($\hat{z}$).
"Donut" shape in 3D

$\langle S \rangle \sim \omega^4$ ← Rayleigh scattering, reason why sky is blue, sunset is red, etc.