PHYS 100C, Lecture 13

Monday, April 26, 2010
6:30 AM

* Potential & Fields

in electrostatics:

$$\nabla \cdot E = \frac{\rho}{\varepsilon_0} \qquad \nabla \times E = -\frac{\partial B}{\partial t} = 0 \quad (B = \text{const})$$

This allowed introducing scalar potential V,

$$\vec{E} = -\vec{\nabla} V$$

Since $\bar{\nabla} \times \bar{E} = -\bar{\nabla} \times (\bar{\nabla} V) = 0$

No longer works if $\partial B / \partial t \neq 0$

$\bar{\nabla} \cdot \bar{B} = 0$ so vector potential still works:

$$\bar{B} = \bar{\nabla} \times \bar{A}$$

then $\nabla \times E = -\frac{\partial B}{\partial t}$ becomes:

$$\nabla \times E = -\frac{\partial}{\partial t}(\nabla \times A) \quad \text{or}$$

$$\nabla \times \left(E + \frac{\partial A}{\partial t}\right) = 0$$

New quantity, $E' = E + \frac{\partial A}{\partial t} = -\nabla V$

since $\nabla \times E' = 0$

then $E = -\nabla V - \frac{\partial A}{\partial t}$

Gauss' Law: $\nabla^2 V + \frac{\partial}{\partial t}(\nabla \cdot A) = -\frac{\rho}{\varepsilon_0}$

similar to Poisson Eq. except for this

last Maxwell Eq:

$$\left(\nabla^2 A - \mu_0 \varepsilon_0 \frac{\partial^2}{\partial t^2}\right) - \nabla\left(\bar{\nabla} \cdot \bar{A} + \mu_0 \varepsilon_0 \frac{\partial V}{\partial t}\right) = -\mu_0 J$$

Monstrous, UGLY equation.

But, reduced 6 unknowns E_x, E_y, E_z, B_x, B_y, B_z to only 4 (V, A_x, A_y, A_z).

Useful for other purposes.

A & V not unique:

can introduce new

$$A' = A + \alpha$$
$$V' = V + \beta$$

that give rise to the same $\vec{E}(\vec{r}, t)$, $\vec{B}(\vec{r}, t)$

Called GAUGE TRANSFORMATIONS

* **Gauge Transform.**

$$\left.\begin{array}{l} A' = A + \alpha \\ V' = V + \beta \end{array}\right\} \text{give the same } E, B \text{ as } A, V$$

$$B = \nabla \times A = \nabla \times A' \Rightarrow \nabla \times \alpha = 0$$

$\alpha = \nabla \lambda$ where λ is scalar

$$E = -\nabla V - \frac{\partial A}{\partial t} = -\nabla V' - \frac{\partial A'}{\partial t} \Rightarrow$$

$$\nabla \beta + \frac{\partial \alpha}{\partial t} = 0 \quad \text{OR} \quad \alpha = \nabla \lambda$$

$$\nabla \left(\beta + \frac{\partial \lambda}{\partial t}\right) = 0$$

Solution: $\beta = -\frac{\partial \lambda}{\partial t}$

$$\left.\begin{array}{l} A' = A + \nabla \lambda \\ V' = V - \frac{\partial \lambda}{\partial t} \end{array}\right\} \text{Give identical } E, B \text{ for any scalar field } \lambda$$

Gauge transformations

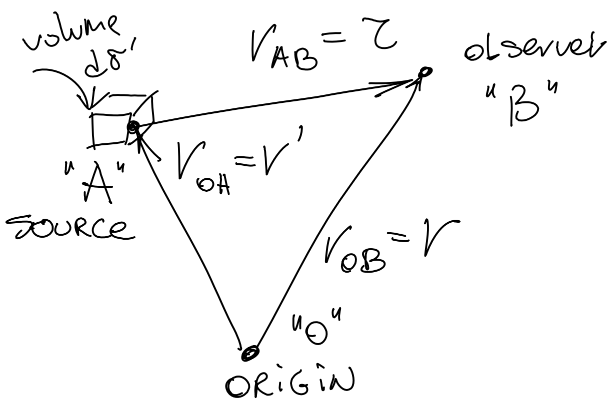
* Coulomb Gauge:

$$\nabla \cdot A = 0 \Rightarrow \nabla^2 V = -\frac{\rho}{\varepsilon_0}$$

Solution is

$$V(r,t) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(r',t)}{\tau}\, d\tau'$$

* Note: three r's:



$$V(r_{OB},t) = \ldots \int \frac{\rho(r_{OA},t)}{r_{OB}}\, d\tau$$

Physics problem with Coulomb gauge:

V is instantaneous (depends on "right now" charge density $\rho(r_{OA},t)$, far away).

E is "retarded" or delayed, by: $E = -\nabla V - \underbrace{\frac{\partial A}{\partial t}}_{\text{this part}}$

* For Superior, AwesomER Gauge:

LORENTZ GAUGE:

$$\nabla\cdot A = -\mu_0\varepsilon_0 \frac{\partial V}{\partial t}$$

to kill of the ugly 2^{nd} part in:

$$\Big(\nabla^2 A - \underbrace{\mu_0\varepsilon_0}_{1/c^2}\frac{\partial^2 A}{\partial t^2}\Big) - \nabla\underbrace{\Big(\nabla\cdot A + \mu_0\varepsilon_0\frac{\partial V}{\partial t}\Big)}_{=0!} = -\mu_0 J$$

Remaining part:

$$\Big(\nabla^2 - \frac{1}{c^2}\cdot\frac{\partial^2}{\partial t^2}\Big) A = -\mu_0 J$$

and $\nabla^2 V + \frac{\partial}{\partial t}(\nabla\cdot A) = -\frac{\rho}{\varepsilon_0}$ becomes

$$\Big(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t}\Big) V = -\frac{\rho}{\varepsilon_0}$$

Introduce d'Alembertian:

$$\Box^2 = \nabla^2 - \frac{1}{c^2}\cdot\frac{\partial^2}{\partial t^2} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial (ict)^2}$$

("ict" is complex four-dimensional space-time coordinate)

Maxwell's Eq's (1) and (4) reduced:

$$\Box^2 V = -\frac{\rho}{\varepsilon_0}$$

$$\Box^2 A = -\mu_0 J$$

(Eqs (2), (3) are already satisfied)

it takes time for light to travel distance r:

potentials evaluated at "retarded" time $t_r \equiv t - \frac{\mathscr{r}}{c}$

$$V(r,t) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(r', t_r)}{\mathscr{r}}\, d\tau'$$

$$A(r,t) = \frac{\mu_0}{4\pi}\int \frac{J(r', t_r)}{\mathscr{r}}\, d\tau'$$

* does NOT work for $E(r,t)$ or $B(r,t)$ – by simply integrating Coulomb's & Biot-Savart's Laws evaluated at time $t_r = t - \mathscr{r}/c$