Physics 100C, Lecture 4:

Thursday, April 09, 2009
10:01 AM

EM Waves in Conductors

$\vec{J} = \sigma \vec{E}$ (Ohm's Law)

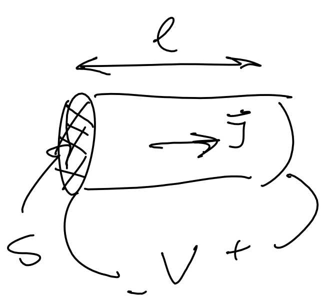
$\vec{J}$ is current density

σ is conductivity

* Sanity check: traditional form of Ohm's Law $I \cdot R = V$

$I = J \cdot S$ (S is area)

$V = E \cdot \ell$ (ℓ is length)



$R = \rho \frac{\ell}{S} = \frac{1}{\sigma} \cdot \frac{\ell}{S}$

$J \cdot \cancel{S} \cdot \frac{1}{\sigma} \cdot \frac{\cancel{\ell}}{\cancel{S}} = E \cdot \cancel{\ell}$

or:

$J = \sigma E$

Maxwell's Eq's, linear media, with free charges and currents:

$$\vec{\nabla} \cdot \vec{E} = \frac{1}{\varepsilon} \rho \qquad (i)$$

$$\vec{\nabla} \cdot \vec{B} = 0 \qquad (ii)$$

$$\vec{\nabla} \times \vec{E} = -\partial B / \partial t \qquad (iii)$$

$$\vec{\nabla} \times \vec{B} = \mu \sigma \vec{E} + \mu \varepsilon \frac{\partial \vec{E}}{\partial t} \qquad (iv)$$

(we substituted $\vec{J} = \sigma \vec{E}$ in (iv))

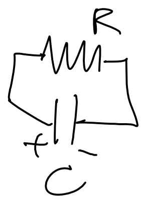
Conservation of charge $\Rightarrow$ similar to Gauss' law, flux of current outside closed volume is equal to $-\frac{\partial \rho}{\partial t}$:

$$\vec{\nabla} \cdot \vec{J} = -\frac{\partial \rho}{\partial t}, \quad \text{or}$$

$$\frac{\partial \rho}{\partial t} = -\vec{\nabla} \cdot \vec{J} = -\sigma \cdot \underbrace{(\vec{\nabla} \cdot \vec{E})}_{\text{Ohm's Law}} = -\underbrace{\frac{\sigma}{\varepsilon} \rho}_{\text{Maxwell (i)}}$$

Solution: $\rho = \rho_0 \cdot \exp\left(-\frac{\sigma}{\varepsilon} t\right)$

Free charge is conducting media "dissipates" (runs to the edges / leaks away) exponentially with characteristic time constant $\tau = \frac{\varepsilon}{\sigma}$



$\Rightarrow Q = Q_0 \cdot e^{-t/RC}$ similar to RC circuit

except media is both R and C

$R \sim \frac{1}{\sigma} \quad C \sim \varepsilon$

"good" conductor $\tau \equiv \frac{\varepsilon}{\sigma} \ll \frac{1}{\omega}$

"bad" conductor $\tau \gg \frac{1}{\omega}$

"perfect" conductor $\sigma \to \infty$
$\tau \to 0$

If we wait $\gg \tau$ for free charge to "dissipate",

$\rho \to 0$ in Eq. (i).

Just like in Lecture #1 Curl (iii) and (iv) gives:

$$\nabla^2 \vec{E} = \mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2} + \mu\sigma \frac{\partial \vec{E}}{\partial t}$$

$$\nabla^2 \vec{B} = \mu\varepsilon \frac{\partial^2 \vec{B}}{\partial t^2} + \mu\sigma \cdot \frac{\partial \vec{B}}{\partial t}$$

this is new!

consequence of allowing $\vec{J} \neq 0$ (free currents)

General Solution is still a planar wave: (along z dir.)

$$\vec{E}(z,t) = \tilde{\vec{E}}_0 \, e^{i(\tilde{k}z - \omega t)}$$

$$\vec{B}(z,t) = \tilde{\vec{B}}_0 \, e^{i(\tilde{k}z - \omega t)}$$

just like for non-conduct. media, but now $\tilde{k}$ is a complex number:

$$\nabla^2 \vec{E} = -\tilde{k}^2 \cdot \vec{E}$$

$$\frac{\partial \vec{E}}{\partial t} = -i\omega \vec{E} \qquad \text{(same for } \vec{B}\text{)}$$

$$\frac{\partial^2 \vec{E}}{\partial t^2} = -\omega^2 \vec{E}$$

New "wave eqs" above become:

$$\tilde{k}^2 = \mu\varepsilon\omega^2 + \underbrace{i\mu\sigma\omega}_{\text{New Part}}$$

Without currents there would be no $i\mu\sigma\omega$ part (think $\sigma \to 0$, insulator) and $\tilde{k}$ is real $\frac{\omega}{k} = v_1 = \frac{1}{\sqrt{\mu\varepsilon}}$

Now $\tilde{k}$ is complex:

$$\tilde{k} = \underbrace{k}_{\text{real}} + \underbrace{i\gamma}_{\text{IMAGINARY}} \qquad \text{OR}$$

$$\tilde{k}^2 = k^2 + 2ik\gamma - \gamma^2 = \mu\varepsilon\omega^2 + i\mu\sigma\omega$$

$$\left.\begin{aligned} k^2 - \gamma^2 &= \mu\varepsilon\omega^2 \\ 2k\gamma &= \mu\sigma\omega \end{aligned}\right\} \text{solve for } k, \gamma:$$

$$k = \omega\sqrt{\frac{\varepsilon\mu}{2}}\left[\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2}+1\right]^{1/2}$$

$$\gamma = \omega\sqrt{\frac{\varepsilon\mu}{2}}\left[\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2}-1\right]^{1/2}$$

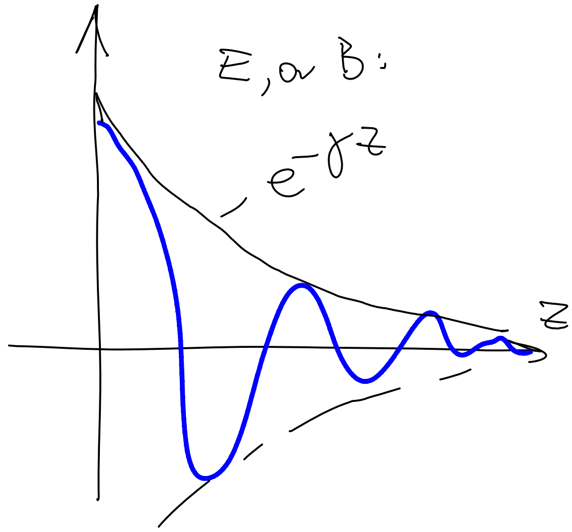
General Solutions become:

$$\vec{E}(z,t) = \tilde{\vec{E}}_0 \cdot e^{-\gamma z} \cdot e^{i(kz-\omega t)}$$

$$\vec{B}(z,t) = \tilde{\vec{B}}_0 \cdot e^{-\gamma z} \cdot e^{i(kz-\omega t)}$$

Both $|\vec{E}|$ and $|\vec{B}|$ are decaying expon. as a function of z, with decay "skin depth"

$$d \equiv \frac{1}{\gamma} \quad (\sim 10\,\text{nm for good metals})$$



$\times$ Check back to Maxwell Eq's.

$$(i) \Rightarrow \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = 0$$

Since we assume no x-y dependence (as in page 5 of Lecture #1 notes)

$$\frac{\partial E_x}{\partial x} = \frac{\partial E_y}{\partial y} = 0 \Rightarrow \frac{\partial E_z}{\partial z} = i\tilde{k}E_z = 0$$

or $E_z = 0$. Same for B (ii)

Waves are still transverse!

Let's define $\hat{x}$ direction so that $\vec{E} \parallel \vec{x}$

$$\vec{E}(z,t) = \tilde{E}_0 e^{-\gamma z} \cdot e^{i(kz-\omega t)} \hat{x}$$

From (iii):

0, since E has no y-dep.

$$\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \tilde{E}_x & 0 & 0 \end{vmatrix} = \hat{y}\frac{\partial \tilde{E}_x}{\partial z} - \hat{z}\cancel{\frac{\partial E_x}{\partial y}} = \hat{y}\frac{\partial E_x}{\partial z} = -\frac{\partial \vec{B}}{\partial t}$$

If we assume B solution depends on time $B \sim e^{-i\omega t}$, then depends on time $B \sim e^{-i\omega t}$, then

$$\frac{\partial \vec{B}}{\partial t} = -i\omega \vec{B}$$

Also, $\frac{\partial \tilde{E}_x}{\partial z} = (ik-\gamma)\vec{E} = i\tilde{k}\vec{E}$

Therefore from (iii) above:

$$\vec{B} = \frac{\tilde{k}}{\omega}\tilde{E}_0 \cdot e^{-\gamma z} \cdot e^{i(kz-\omega t)}\hat{y}$$

So $\vec{E} \perp \vec{B}$, and $\vec{E}, \vec{B} \perp \vec{k}$ (along $\hat{z}$) just like before.

But! $|\vec{E}|$ and $|\vec{B}|$ are out of phase, since $\tilde{k}$ is complex!

if $\tilde{k} = \sqrt{k^2+\gamma^2} \cdot e^{i\varphi}$, then:

$$\vec{E}(z,t) = E_0 e^{-\gamma z} \cdot \cos(kz - \omega t + \delta_E)\hat{x}$$

$$\vec{B}(z,t) = \frac{E_0}{\omega}\sqrt{k^2+\gamma^2} \cdot e^{-\gamma z} \cdot \cos(k_z - \omega t + \delta_E + \varphi)\hat{y}$$

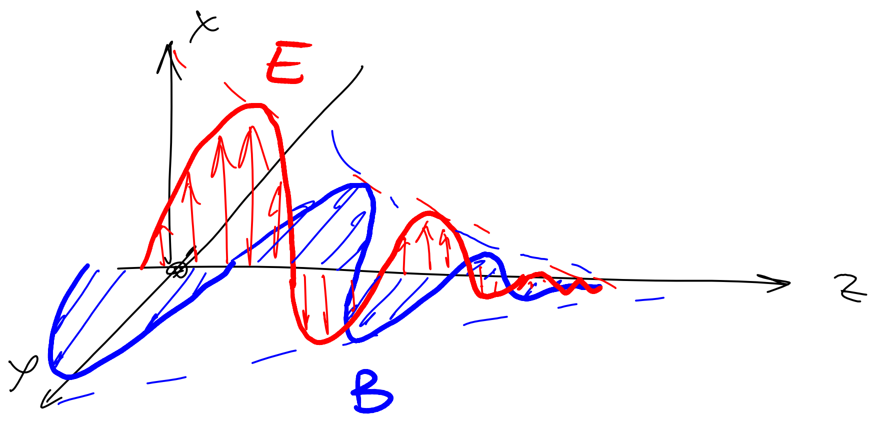
E_0 is now REAL $\left(\tilde{E}_0 = E_0 \cdot e^{i\delta_E}\right)$

Before we had the same phase for $\vec{E}$ and $\vec{B}$ ($\delta_E = \delta_B$)

which we "swept under the rug" by offsetting $t=0$ by $\Delta t = \frac{\delta_E}{\omega}$

Now we have additional phase φ, where:

$\varphi = \arctan\left(\frac{\gamma}{k}\right)$ (just like Prob 9.3)



* Reflection/transmission at conducting interface

So what happened to EM wave if it can't get through? It gets reflected!

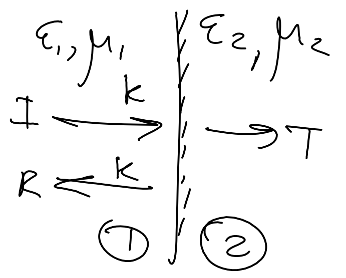
Boundary conditions:

$\varepsilon_1 E_1^\perp - \varepsilon_2 E_2^\perp = \sigma$ (i)

$B_1^\perp - B_2^\perp = 0$ (ii)

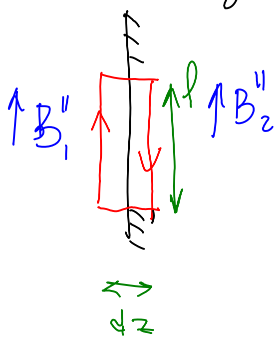
$E_1^\parallel - E_2^\parallel = 0$ (iii)

$\dfrac{B_1^\parallel}{\mu_1} - \dfrac{B_2^\parallel}{\mu_2} = \vec{K} \times \vec{n}$ (iv)



For conductors $\vec{K}$ (surface current per unit length) must be equal to 0, otherwise J (current per area) $\to \infty$ and $E = \dfrac{J}{\sigma} \to \infty$ as well

*(We can have finite current densities J, but then total current per unit length can be made $\to 0$ by tightening the Ampere's Loop



$dz \to 0$, $|\vec{K}| = J \cdot dz \cdot \ell \to 0$)

Just like in Lectures 2,3:

Incident wave:

$$E_I = \tilde{E}_{0I} e^{i(k_1 z - \omega t)}; \quad B_I = \frac{\tilde{E}_{0I}}{v_1} e^{i(k_1 z - \omega t)}$$

Reflected:

$$E_R = \tilde{E}_{0R} \cdot e^{i(-k_1 z - \omega t)}; \quad B_R = -\frac{\tilde{E}_{0R}}{v_1} e^{i(-k_1 z - \omega t)}$$

Transmitted wave has the solution derived above:

$$E_T = \tilde{E}_{0T} e^{i(\tilde{k}_2 z - \omega t)}; \quad B_R = \frac{\tilde{k}_2 \tilde{E}_{0T}}{\omega} e^{i(\tilde{k}_2 z - \omega t)}$$

Same as before, but $\tilde{k}$ is complex.

Boundary conditions:

(i) & (ii) yield nothing

since $E_\perp = B_\perp = 0$ (ALSO means $\sigma = 0$)

(iii): $\tilde{E}_{0I} + \tilde{E}_{0R} = \tilde{E}_{0T}$

(iv): $\dfrac{\tilde{E}_{0I}}{\mu_1 v_1} - \dfrac{\tilde{E}_{0R}}{\mu_1 v_1} = \dfrac{\tilde{k}_2}{\mu_2 \omega} \tilde{E}_{0T}$

Solutions are:

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$$\tilde{E}_{0R} = \left(\frac{1-\tilde{\beta}}{1+\tilde{\beta}}\right)\tilde{E}_{0I}$$

$$\tilde{E}_{0T} = \left(\frac{2}{1+\tilde{\beta}}\right)\tilde{E}_{0I}$$

(Same as before but $\tilde{\beta} = \frac{\mu_1 v_1}{\mu_2 \omega}\tilde{k}_2$ is complex now!)

Perfect conductor $\sigma \to \infty$

$\Rightarrow \tilde{k}_2 \to \infty$

$\Rightarrow \tilde{\beta} \to \infty$

and

$$\tilde{E}_{0R} = -\tilde{E}_{0I} \quad ; \quad \tilde{E}_{0T} = 0$$

$$R \equiv \left(\frac{|\tilde{E}_{0R}|}{|\tilde{E}_{0I}|}\right)^2 = 1 \qquad T = 0$$

100% reflection at normal incidence, reflected wave is 180° out of phase w.r. to incident.

Please verify this every day by looking at the mirror reflection (also check for vampires!)

* PHYSICS SUMMARY:

EM waves in conductors (metals) induce currents (confined to "skin depth" near the surface) which "screen" the wave from passing through conductor.

As a result, waves get reflected (primarily), giving rise to shiny objects, mirrors etc.