

**Problem 9.13** Calculate the *exact* reflection and transmission coefficients, *without* assuming  $\mu_1 = \mu_2 = \mu_0$ . Confirm that  $R + T = 1$ .

**Problem 9.16** Analyze the case of polarization *perpendicular* to the plane of incidence (i.e. electric fields in the  $y$  direction, in Fig. 9.15). Impose the boundary conditions 9.101, and obtain the Fresnel equations for  $\tilde{E}_{0R}$  and  $\tilde{E}_{0T}$ . Sketch  $(\tilde{E}_{0R}/\tilde{E}_{0I})$  and  $(\tilde{E}_{0T}/\tilde{E}_{0I})$  as functions of  $\theta_I$ , for the case  $\beta = n_2/n_1 = 1.5$ . (Note that for this  $\beta$  the reflected wave is *always*  $180^\circ$  out of phase.) Show that there is no Brewster's angle for *any*  $n_1$  and  $n_2$ :  $\tilde{E}_{0R}$  is *never* zero (unless, of course,  $n_1 = n_2$  and  $\mu_1 = \mu_2$ , in which case the two media are optically indistinguishable). Confirm that your Fresnel equations reduce to the proper forms at normal incidence. Compute the reflection and transmission coefficients, and check that they add up to 1.

**Problem 9.19**

(a) Show that the skin depth in a poor conductor ( $\sigma \ll \omega\epsilon$ ) is  $(2/\sigma)\sqrt{\epsilon/\mu}$  (independent of frequency). Find the skin depth (in meters) for (pure) water.

(b) Show that the skin depth in a good conductor ( $\sigma \gg \omega\epsilon$ ) is  $\lambda/2\pi$  (where  $\lambda$  is the wavelength *in the conductor*). Find the skin depth (in nanometers) for a typical metal ( $\sigma \approx 10^7 (\Omega \cdot \text{m})^{-1}$ ) in the visible range ( $\omega \approx 10^{15}/\text{s}$ ), assuming  $\epsilon \approx \epsilon_0$  and  $\mu \approx \mu_0$ . Why are metals opaque?

(c) Show that in a good conductor the magnetic field lags the electric field by  $45^\circ$ , and find the ratio of their amplitudes. For a numerical example, use the "typical metal" in part (b).

**Problem 9.21** Calculate the reflection coefficient for light at an air-to-silver interface ( $\mu_1 = \mu_2 = \mu_0$ ,  $\epsilon_1 = \epsilon_0$ ,  $\sigma = 6 \times 10^7 (\Omega \cdot \text{m})^{-1}$ ), at optical frequencies ( $\omega = 4 \times 10^{15}/\text{s}$ ).

9.13

$$E_{0r} = \frac{1-\beta}{1+\beta} E_{0I}$$

$$\beta = \frac{\mu_1 v_1}{\mu_2 v_2} \left( = \frac{\epsilon_2 v_2}{\epsilon_1 v_1} \right)$$

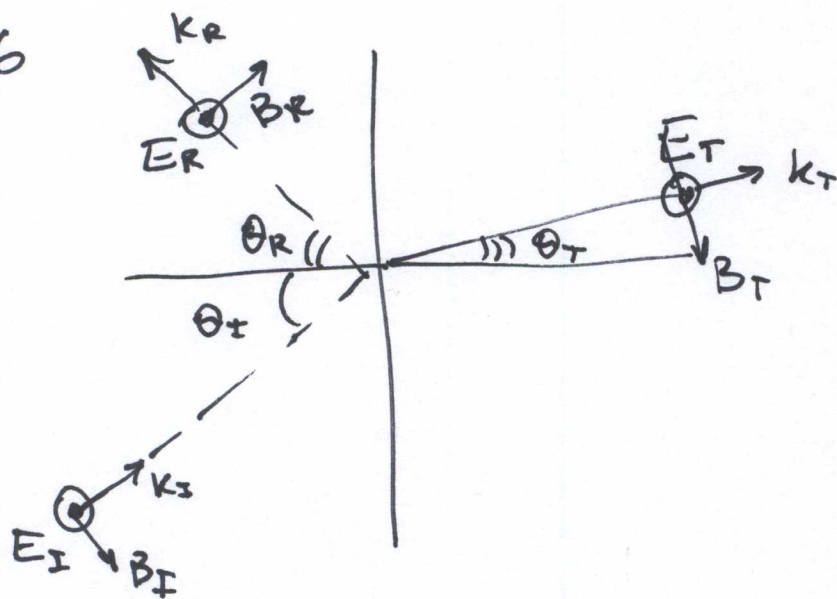
$$R = \left( \frac{E_{0r}}{E_{0I}} \right)^2 = \left( \frac{1-\beta}{1+\beta} \right)^2$$

$$E_{0T} = \left( \frac{2}{1+\beta} \right) E_{0I}$$

$$T = \frac{I_T}{I_I} = \left( \frac{E_{0T}}{E_{0I}} \right)^2 \cdot \frac{\epsilon_2 v_2}{\epsilon_1 v_1} = \frac{4}{(1+\beta)^2} \cdot \beta$$

$$R+T = \frac{(1-\beta)^2 + 4\beta}{(1+\beta)^2} = \frac{1 - 2\beta + \beta^2 + 4\beta}{(1+\beta)^2} = \frac{(1+\beta)^2}{(1+\beta)^2} = 1$$

Q. 16



BOUNDARY CONDITIONS:

(i)  $D_{\perp} = \epsilon E_{\perp}$       $0 = 0$

(ii)  $B_{\perp}$ :  $B_{0I} \sin \theta_I + B_R \sin \theta_R = B_T \sin \theta_T$

OR, since  $B = \frac{E}{v}$ :  $E_{0I} + E_{OR} = \frac{v_1 \sin \theta_T}{v_2 \sin \theta_I} \cdot E_{OT}$   
 $\theta_R = \theta_I$

OR, since Snell's law  $\frac{\sin \theta_T}{\sin \theta_I} = \frac{n_1}{n_2} = \frac{v_2}{v_1}$ ,

$E_{0I} + E_{OR} = E_{OT}$

(iii)  $E_{\parallel}$ :  $E_{0I} + E_{OR} = E_{OT}$  (SAME AS (ii))

(iv):  $\frac{1}{\mu_1} \left( -\frac{E_{0I}}{v_1} \cos \theta_I + \frac{E_{OR}}{v_2} \cos \theta_R \right) = \frac{1}{\mu_2} \left( -\frac{E_{OT}}{v_2} \right) \cos \theta_T$

$E_{0I} - E_{OR} = E_{OT} \cdot \frac{\mu_1 v_1}{\mu_2 v_2} \cdot \frac{\cos \theta_T}{\cos \theta_I}$

If as before,  $\alpha \equiv \frac{\cos \theta_T}{\cos \theta_I}$ ,  $\beta \equiv \frac{\mu_1 v_1}{\mu_2 v_2} \Rightarrow E_{0I} - E_{OR} = E_{OT} \alpha \beta$

9.16, cont'd:  $E_{OT} = \frac{2}{1+\alpha\beta} E_{OI}$

$$E_{OR} = E_{OT} - E_{OI} = \left( \frac{2}{1+\alpha\beta} - 1 \right) E_{OI} = \frac{1-\alpha\beta}{1+\alpha\beta} E_{OI}$$

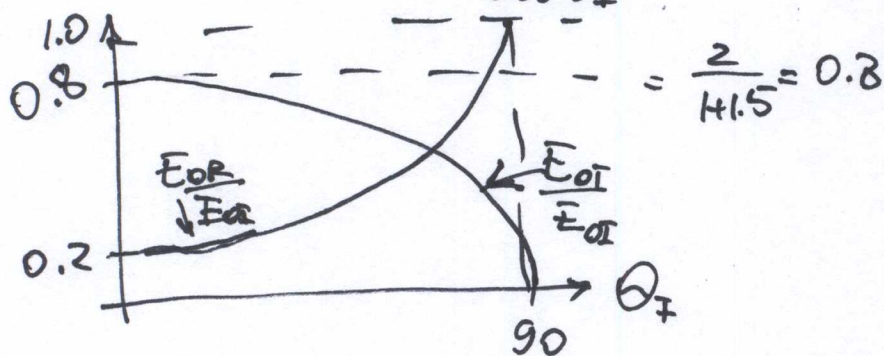
$E_{OT}$  is always in-phase with  $E_{OI}$  ( $\frac{2}{1+\alpha\beta} > 0$ )

$E_{OR}$  is in-phase with  $E_{OI}$  if  $\alpha\beta < 1$   
out-of-phase by  $180^\circ$  if  $\alpha\beta > 1$

$$\alpha \cdot \beta = \beta \cdot \frac{\sqrt{1 - \sin^2 \theta_T}}{\cos \theta_I} = \beta \frac{\sqrt{1 - \sin^2 \theta_I \left( \frac{n_1}{n_2} \right)^2}}{\cos \theta_I}$$

or, since  $\mu_1 \approx \mu_2$   $\frac{n_1}{n_2} \approx \frac{1}{\beta}$

$$\alpha \cdot \beta = \beta \frac{\sqrt{1 - \frac{\sin^2 \theta_I}{\beta^2}}}{\cos \theta_I} = \frac{\sqrt{\beta^2 - \sin^2 \theta}}{\cos \theta} > 0$$



Brewster angle?  $E_{OR} = 0$  if  $\alpha\beta = 1$

$$\text{or } \frac{\beta^2 - \sin^2 \theta_I}{\cos^2 \theta_I} = 1 \Rightarrow \beta^2 = \sin^2 \theta_I + \cos^2 \theta_I = 1$$

But  $\beta = 1.5 > 1$

For NORMAL incidence  $\theta_I = 0$   $\alpha = 1$  and

$$E_{OR} = \frac{1-\beta}{1+\beta} E_{OI} \quad E_{OT} = \frac{2}{1+\beta} E_{OI}$$

$$R \equiv \left( \frac{E_{OR}}{E_{OI}} \right)^2 = \left( \frac{1-\alpha\beta}{1+\alpha\beta} \right)^2 \quad T \equiv \frac{\epsilon_2 v_2}{\epsilon_1 v_1} \cdot \alpha \left( \frac{E_{OT}}{E_{OI}} \right)^2 = \alpha \beta \left( \frac{2}{\alpha+\beta} \right)^2$$

$$R+T = \frac{1-2\alpha\beta + \alpha^2\beta^2 + 4\alpha\beta}{(\alpha+\beta)^2} = \frac{(\alpha+\beta)^2}{(\alpha+\beta)^2} = 1$$

9.19

$$k = \omega \sqrt{\frac{\epsilon \mu}{2}} \left[ \sqrt{1 + \left(\frac{\sigma}{\epsilon \omega}\right)^2} + 1 \right]^{1/2}$$

$$\gamma = \omega \sqrt{\frac{\epsilon \mu}{2}} \left[ \sqrt{1 + \left(\frac{\sigma}{\epsilon \omega}\right)^2} - 1 \right]^{1/2}$$

A) Poor conductors  $x \equiv \frac{\sigma}{\epsilon \omega} \ll 1$      $\sqrt{1+x^2} \approx 1 + \frac{x^2}{2}$

$$\gamma \approx \omega \sqrt{\frac{\epsilon \mu}{2}} \cdot \left(\frac{\sigma}{2\epsilon \omega}\right) = \sqrt{\frac{\mu}{2\epsilon}} \cdot \sigma = \frac{\sigma}{2} \sqrt{\frac{\mu}{\epsilon}}$$

$$\text{Skin depth } d \equiv \frac{1}{\gamma} = \frac{2}{\sigma} \cdot \sqrt{\frac{\epsilon}{\mu}}$$

For water  $\mu \approx \mu_0$      $\epsilon = 80\epsilon_0$      $\sigma = 4 \cdot 10^{-6}$

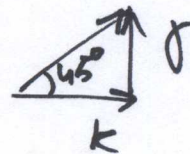
$$d = 1.2 \cdot 10^4 \text{ m OR } 12 \text{ km}$$

B) Good conductor  $\frac{\sigma}{\epsilon \omega} \gg 1$

$$k \approx \gamma \approx \omega \sqrt{\frac{\epsilon \mu}{2}} \cdot \sqrt{\frac{\sigma}{\epsilon \omega}} = \sqrt{\frac{\sigma \mu \omega}{2}}$$

$$d \equiv \frac{1}{\gamma} \approx \sqrt{\frac{2}{\sigma \mu \omega}} = \sqrt{\frac{2}{10^{15} \cdot 1.2 \cdot 10^{-6} \cdot 10^7}} = 1.3 \cdot 10^{-8} \text{ m OR } 13 \text{ nm}$$

$$k \approx \gamma \quad \phi = \tan^{-1}(1) = 45^\circ$$



$$\frac{B_0}{E_0} = \frac{\sqrt{k^2 + \gamma^2}}{\omega} = \sqrt{\frac{\sigma \mu}{\omega}} = 10^{-7} \frac{\text{S}}{\text{m}}$$

$$9.21 \quad R = \left( \frac{E_{OR}}{E_{OT}} \right)^2 = \left| \frac{1 - \tilde{\beta}}{1 + \tilde{\beta}} \right|^2$$

$$\tilde{\beta} = \frac{\mu_1 v_1}{\mu_2 \omega} \tilde{k}$$

$$\tilde{k} = k + ik$$

(since  $\sigma \approx k$  for good conductors,  
see 9.19)

$$k = \sqrt{\frac{\sigma \mu \omega}{2}} \quad \text{---}$$

$$\frac{\mu_1 v_1}{\mu_2 \omega} \cdot k \approx \sqrt{\frac{\sigma \mu_0}{2 \omega}} \cdot c \approx 29$$

$$R = \left| \frac{1 - 29 - i \cdot 29}{1 + 29 + i \cdot 29} \right|^2 = \frac{28^2 + 29^2}{30^2 + 29^2} = 0.933 \quad \text{or} \\ \approx 93\%$$